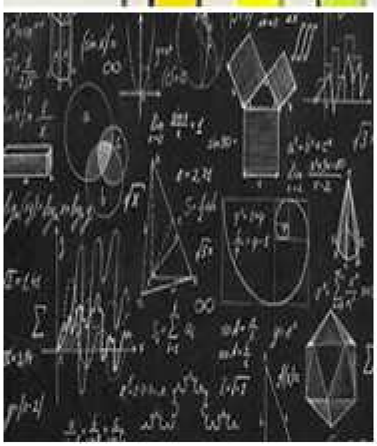
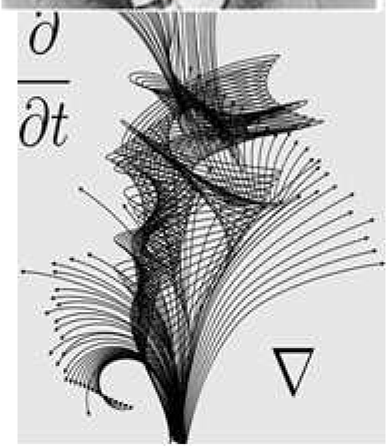
Master of Science
(SECOND SEMESTER)

MAT508
ADVANCED DIFFERENTIAL EQUATIONS-II







DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME

ADVANCED DIFFERENTIAL EQUATIONS-II

COURSE CODE: MAT 508

Department of Mathematics

School of Science

Uttarakhand Open University

Haldwani, Uttarakhand, India,

263139

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Editor

Dr. Kamlesh Bisht
Assistant Professor(AC)
Department of Mathematics,
School of Sciences, Uttarakhand Open University

Unit Writer	Block	Unit
Dr. Jyoti Rani Assistant Professor Department of Mathematics Uttarakhand Open University Haldwani, Nainital	**I,II,III, IV**	01 to 14

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MAT 508

COURSE INFORMATION

The present self learning material **"Advanced Differential Equations-II"** has been designed for M.Sc. (Second Semester) learners of Uttarkhand Open University, Haldwani. This self learning material is writing for increase learner access to high-quality learning materials.This course is divided into 14 units of study. The first five units are devoted to **Block-I** (**Advanced Differential Equations-I).** Typically builds upon the foundations of Advanced Differential Equations-I, exploring more advanced topics such as nonlinear differential equations and advanced applications in physics, engineering, etc., and other fields. **Block-II (Advanced Differential Equations-II)** typically covers topics beyond introductory differential equations, delving into more complex equations, techniques, and applications. This might include methods like Fourier series, Laplace transforms, boundary value problems, and partial differential equations. **Block-III (Advanced Differential Equations-III)** usually covers even more specialized topics, such as advanced techniques in solving partial differential equations (PDEs), including numerical methods, Green's functions, variational methods, and advanced applications in areas. **Block-IV(Numerical solution of PDEs)** involves using computational methods to approximate solutions to partial differential equations. This typically includes discretizing the spatial and temporal domains, employing numerical techniques such as finite difference, finite element, or spectral methods, and solving resulting algebraic equations iteratively to approximate the PDE solution.

SYLLABUS

Advanced Differential Equation-II **Course code: MAT508**

Credit:04

Partial Differential Equations-I: Formation and Solutions of PDE, Complete Integral, General solution of Lagrange Equation, working rule (Example based), Surface and normal's in three Dimensions, Curve in three dimensions intersection of two surfaces. Integral Surfaces passing through a given curve (The Cauchy Problem), Surface orthogonal to a given system of surfaces, Geometrical description of Lagrange's equation $Pp + Qq = R$ and Lagrange's auxiliary equations $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$ Geometrical interpretation of $Pp + Qq = R$, Linear Partial Differential Equations of order one with n independent variables , Non-Linear Partial Differential Equations of order one, Fundamentals: Classification and Canonical Forms of PDE, Monge's Method

Partial Differential Equations-II: Derivation of Laplace and Poisson Equation, Dirichlet's Problem and Newmann Problem for a Rectangle, Interior and Exterior Dirichlet's Problem for a circle, Interior Newmann problem for a circle, Solution of Laplace equation in Cylindrical and spherical coordinates, Examples. Formation and Solution of diffusion Equation.

Partial Differential Equation-III: Dirac-Delta function, Separation of Variables Method, Solution of Diffusion Equation in Cylindrical and Spherical Coordinates, Examples. Formation and Solution of One-Dimensional wave equation, Canonical Reduction, D' Alembert's Solution, Two- Dimensional Wave Equation, Periodic Solution of One- Dimensional Wave Equation in Cylindrical and Spherical Coordinate Systems, Uniqueness of the Solution for the Wave Equation, Examples. Green's function for Laplace equation, Methods of Images, Eigen Function Method, Green's Function for the Wave and Diffusion Equations. Laplace Transform method: Solution of Diffusion and Wave equation by Laplace Transform.

Numerical Solutions of PDEs: Finite differences of Partial Differential Equations(PDEs), Applications to Integral Equations, Finite Element Method.

References:

1. Earl A. Coddington (1961).An Introduction to Ordinary Differential Equations, Dover Publications.
2. Lawrence C. Evans (2010).Partial Differential Equations. (2^{nd} edition).American Mathematical Society.
3. M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20^{th} Edition), S. Chand.
4. K.S. Rao, (2011). Introduction to Partial Differential Equations (3^{rd} edition), Prentice Hall India Learning Private Limited,

Suggested Readings:

1. Erwin Kreyszig (2011). Advanced Engineering Mathematics (10th edition). Wiley.
2. B. Rai, D. P. Choudhury & H. I. Freedman (2013). A Course in Ordinary Differential Equations (2nd edition). Narosa.
3. Shepley L. Ross (2007). Differential Equations (3rd edition), Wiley India.
4. George F. Simmons (2017). Differential Equations with Applications and Historical Notes (3rd edition). CRC Press. Taylor & Francis.

BLOCK I
PARTIAL DIFFERENTIAL EQUATIONS-I

Unit 1: Formation and Solution of PDEs

CONTENTS:

1.1 *INTRODUCTION:-*

Partial Differential Equations (PDEs) play a crucial role in describing and understanding a wide range of physical phenomena and mathematical concepts. They are fundamental tools in fields such as physics, engineering, biology, finance, and more. PDEs describe how functions and variables change with respect to multiple independent variables, including time and space. This introduction provides an overview of the formation and solution of PDEs, highlighting their significance and the approaches used to tackle them.

PDEs are essential mathematical tools for modeling dynamic processes across various fields. Their formation involves translating physical systems into mathematical equations, while their solution requires a combination of analytical and numerical techniques. PDEs provide a bridge between theory and real-world applications, enabling us to make informed decisions and advancements in science and technology.
In this unit, we propose to study various methods to solve partial differential equations.

1.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To develop a fundamental understanding of what partial differential equations are and how they differ from ordinary differential equations.
- To Understand the Basics of PDEs.
- To Analyzing the Physical Phenomena.

The objectives of studying the formation of solutions to PDEs are designed to equip learners with a solid foundation in the theory and application of partial differential equations, preparing them to tackle diverse challenges in mathematics, physics, engineering, and other scientific disciplines.

1.3 *PARTIAL DIFFERENTIAL EQUATION:-*

A Partial Differential Equation (PDE) is a type of differential equation that involves multiple independent variables and their partial derivatives with respect to those variables. Unlike ordinary differential equations (ODEs), which involve a single independent variable, PDEs deal with functions of two or more independent variables.

Or

"*An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as partial differential equation*.**"**

Mathematically, a partial differential equation typically takes the form:

$$F\left(x_1, x_2, \cdots x_n z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \cdots \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1^2}, \frac{\partial^2 z}{\partial x_2^2}, \cdots \frac{\partial^2 z}{\partial x_n^2}\right) = 0 \quad \cdots (1)$$

where

- $x_1, x_2, \cdots x_n$ are the independent variables,
- z is the unknown function of these variables,
- $\frac{\partial z}{\partial x_i}, (i = 1,2 \cdots n)$ represents the partial derivative of u with respect to x_i **(the first-order partial derivative)**.
- $\frac{\partial^2 z}{\partial x_i^2}$ represents the **second-order partial derivative** of u with respect to x_i, and F is some mathematical expression that relates u and its partial derivatives.

1.4 *ORDER OF PARTIAL DIFFERENTIAL EQUATION:-*

The order of a partial differential equation (PDE) refers to the highest order of partial derivatives present in equation (1).

For Example:

- The equations $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy,\quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz,\quad z\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial y}=x$ are of the **first order.**
- The equations $\frac{\partial^2 z}{\partial y^2}=\left(1+\frac{\partial z}{\partial y}\right)^{1/2},\ \left(\frac{\partial z}{\partial x}\right)^2+\frac{\partial^3 z}{\partial y^3}=2x\left(\frac{\partial z}{\partial x}\right)$ are of the **second and third order.**

1.5 *DEGREE OF PARTIAL DIFFERENTIAL EQUATION:-*

The degree of a partial differential equation (PDE) is the highest power to which the highest-order partial derivative term is raised in the equation.

For Example: $a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+du=0$

In this PDE, the highest-order partial derivative term is $\frac{\partial^2 u}{\partial x^2}$, and its degree is 2 because it is raised to the power of 2.

1.6 *LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:-*

The partial differential equation is called ***LINEAR*** if the dependent variable and its partial derivatives occur only in the first degree and not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation.

Example: The equation $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy, \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$ are Linear.

A partial differential equation which is not Linear is known as **NON-LINEAR** partial differential equation.

Example: The equation $z\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial y}=x, \frac{\partial^2 z}{\partial y^2}=\left(1+\frac{\partial z}{\partial y}\right)^{1/2}$ are non-linear.

Notation: Now we adopt the following notations throughout the study of PDEs

$p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x \partial y}$ and $r=\frac{\partial^2 z}{\partial y^2}$

Let we take $x_1, x_2 \dots\dots x_n$(n independent variable) and z is then regarded as the dependent variable. Hence we use the following notation.

$p_1=\frac{\partial z}{\partial x_1}$, $p_2=\frac{\partial z}{\partial x_2}$, $p_3=\frac{\partial z}{\partial x_3}$ and $p_n=\frac{\partial z}{\partial x_n}$

$u_x=\frac{\partial u}{\partial x}$, $u_y=\frac{\partial u}{\partial y}$, $u_{xx}=\frac{\partial^2 u}{\partial x^2}$, $u_{xy}=\frac{\partial^2 u}{\partial x \partial y}$ and so on.

1.7 *CLASSIFICATION OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS:-*

First-order partial differential equations (PDEs) can be classified into four categories: linear, semi-linear, quasi-linear, and non-linear. These classifications are based on the degree of linearity in the PDEs. Here's an explanation of each category with examples:

A first order partial differential equation in two variables in its most general form can be expressed as

$$F(x,y,z,p,q)=0 \qquad \dots(1)$$

Where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$, z is dependent variable and x,y is independent variables.

a. **Linear PDEs:** Linear PDEs are those in which all terms involving the dependent variable and its partial derivatives are of the first degree. OR

A first order partial differential equation $F(x,y,z,p,q)=0$ is said to be ***LINEAR*** if it is linear p,q and z *i.e.,* If given equation is of the form

$$P(x,y)p+Q(x,y)q=R(x,y)z+S(x,y).$$

For example:

$$yx^2p+xy^2q=xyz+x^2y^3$$
$$p\cos(x+y)+q\sin(x+y)=z+e^y\sin x$$
$$p+3q=5z+\tan(y-3x)$$
$$p+q=z+xy \text{ etc.,}$$

are all linear partial differential equations.

b. Semi-Linear PDEs: A first order partial differential equation $F(x, y, z, p, q) = 0$ is said to be ***SEMI-LINEAR*** if it is linear p, q and the coefficient of p and q are the functions of x and y *i.e.,* If it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

For example:

$$(x + y^2)p + xlogy\, q = 2z^2x + xy + e^x$$
$$pcos(x + y) + qsin(x + y) = z^3 + e^y + sinx$$
$$xyp + x^2yq = x^2y^2z^2$$
$$yp + xq = (x^2z^2/y^2) \text{ etc.,}$$

are all Semi-linear partial differential equations.

c. Quasi-Linear PDEs: A first order partial differential equation $F(x, y, z, p, q) = 0$ is said to be ***QUASI-LINEAR*** if it is linear in p, q *i.e.,* If it is of the form

$$P(x, y, z) + Q(x, y, z)q = R(x, y, z)$$

For example:

$$(x + y + z)p + xyz + xz = 3x^2 + 5y^2 + 6z^2$$
$$(x^2 + y^2)p + 4xyzq = 3z + e^{x+y}$$
$$(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy) \text{ etc.,}$$

are all Quasi-linear partial differential equations.

d. Non-Linear PDEs: A first order partial differential equation $F(x, y, z, p, q) = 0$ which does not come under the above three types, called Non-Linear equation.

For example:

$$p^2 + q^2 = 1$$
$$pq = z$$
$$x^2p^2 + y^2q^2 = z^2 \text{ etc.,}$$

are all Non-linear partial differential equations.

1.8 *FORMATION OF PDEs:-*

Partial differential equation can be formed either by elimination of arbitrary constants or by elimination of arbitrary functions.

<u>**RULE1:**</u> **Derivation of a partial differential equation by elimination of arbitrary constants.**

Let us consider $F(x, y, z, p, q) = 0$ ... (1)

where a, b are arbitrary constants. Let z be the function of two independent variables x and y.

Now differentiating (1) w.r.t x and y, we obtain

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0$$
$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0$$

Solving these two equations we can formulate partial differential equation (1).

Situation I: *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

Example: Let $z = ax + y$

Differentiating above equation w.r.t. x and y , we obtain

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = 1$, then

$$z = x\left(\frac{\partial z}{\partial x}\right) + y$$

Situation II: *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants usually gives rise to unique partial differential equation of order one.*

Example: Let $az + b = a^2x + y$

Differentiating above equation w.r.t. x and y , we get

$a\frac{\partial z}{\partial x} = a^2$ and $a\left(\frac{\partial z}{\partial y}\right) = 1$, then

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

Situation III: *When the number of arbitrary constants is grater then the number of independent variables, then the elimination of arbitrary constants leads to unique partial differential equation usually greater than one.*

Example: Let $z = ax + by + cxy$ ... (1)

Differentiating above equation w.r.t. x and y , we obtain

$$\frac{\partial z}{\partial x} = a + cy, \frac{\partial z}{\partial y} = b + cx \quad \text{... (2)}$$

and

$$\frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y^2} = 0, \frac{\partial^2 z}{\partial x \partial y} = c \quad \text{... (3)}$$

then

$\left(\frac{\partial z}{\partial x}\right)x = ax + cxy$ and $\left(\frac{\partial z}{\partial y}\right)y = by + cxy$

Now from (1)

$$z = \left(\frac{\partial z}{\partial x}\right) x - cxy + \left(\frac{\partial z}{\partial y}\right) y - cxy + cxy$$

$$z + cxy = \left(\frac{\partial z}{\partial x}\right) x + \left(\frac{\partial z}{\partial y}\right) y \Rightarrow \; z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \left(\frac{\partial z}{\partial x}\right) x + \left(\frac{\partial z}{\partial y}\right) y.$$

SOLVED EXAMPLE

EXAMPLE1: Solve the partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.

SOLUTION: The given equation is

$$z = ax + by + a^2 + b^2 \qquad \ldots (1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$.

Putting the value of a and b in (1), we have

$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$. Which is required solution (PDEs).

EXAMPLE2: Solve the partial differential equation by eliminating h and k from $(x-h)^2 + (y-k)^2 + z^2 = \lambda^2$.

SOLUTION: The given equation is

$$(x-h)^2 + (y-k)^2 + z^2 = \lambda^2 \qquad \ldots (1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$2(x-h) + 2z\left(\frac{\partial z}{\partial x}\right) = 0$ and $(x-h) = -z\left(\frac{\partial z}{\partial x}\right)$

and

$2(y-k) + 2z\left(\frac{\partial z}{\partial y}\right) = 0$ and $(x-k) = -z\left(\frac{\partial z}{\partial y}\right)$

Putting the value of $(x-h)$ and $(x-k)$in (1), we obtain

$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 + z^2 = \lambda^2$ and $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \lambda^2$

Which is required solution (PDEs).

EXAMPLE3: Solve the partial differential equation by eliminating a and b from the following relations:

a. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
b. $2z = (ax+y)^2 + b$

SOLUTION:

a. Let the equation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \ldots (1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$$2\left(\frac{\partial z}{\partial x}\right) = \frac{2x}{a^2} \;\Rightarrow p = \left(\frac{\partial z}{\partial x}\right) = \frac{x}{a^2} \;\Rightarrow a^2 = \frac{x}{p}$$

$$2\left(\frac{\partial z}{\partial y}\right) = \frac{2y}{b^2} \Rightarrow q = \left(\frac{\partial z}{\partial y}\right) = \frac{y}{b^2} \Rightarrow b^2 = \frac{y}{p}$$

Putting the value of a^2 and b^2in (1), we get

$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{\left(\frac{x}{p}\right)^2} + \frac{y^2}{\left(\frac{y}{p}\right)^2} = px + qy$. is required solution (PDEs).

b. Let the equation

$$2z = (ax + y)^2 + b \qquad \dots (1)$$

Differentiating (1) equation w.r.t. x and y , we have

$$2p = 2a(ax + y) \Rightarrow p = a(ax + y)$$
$$2q = 2(ax + y) \Rightarrow q = (ax + y)$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Dividing both above equation $\frac{p}{q} = a$.

Putting the value of a in (1), we obtain

$q = \left(\frac{p}{q}\right)x + y$ or $px + qy = q^2$.

EXAMPLE4: Solve the partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

SOLUTION: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \dots (1)$

Differentiating (1) equation w.r.t. x and y , we can write

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; c^2x + a^2z\frac{\partial z}{\partial x} = 0 \qquad \dots (2)$$
$$\frac{2x}{b^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \;\Rightarrow\; c^2x + a^2z\frac{dz}{dx} = 0 \qquad \dots (3)$$

Differentiating (2) w.r.t. x and(3) w.r.t. y , we obtain

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \qquad \dots (4)$$
$$c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0 \qquad \dots (5)$$

Now again from (2),

$$c^2x = -a^2z\frac{\partial z}{\partial x}$$
$$c^2 = -\frac{za^2}{x}\frac{\partial z}{\partial x}$$

Substituting the value of c^2 in (4) and dividing by a^2, we obtain

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0$$
$$-\frac{za^2}{xa^2}\frac{\partial z}{\partial x} + \frac{a^2}{a^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{a^2}{a^2}z\frac{\partial^2 z}{\partial x^2} = 0$$

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0 \text{ or } \; zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \quad \dots(7)$$

Similarly from (3) and (5), we get

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \qquad \dots(8)$$

Again differentiating (2) partially w.r.t. y,

$$0+a^2\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0$$

Or

$$a^2\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0 \qquad \dots(9)$$

Hence (7), (8) and (9) are three possible forms of the required PDEs.

EXAMPLE5: Solve the partial differential equation by eliminating a, b, c from $ax^2+by^2+cz^2=1$.

SOLUTION: Given the equation

$$ax^2+by^2+cz^2=1 \qquad \dots(1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$$2ax+2cz\left(\frac{\partial z}{\partial x}\right)=0 \qquad \dots(2)$$

$$2by+2cz\left(\frac{\partial z}{\partial y}\right)=0 \qquad \dots(3)$$

Differentiating (2) equation w.r.t. y , we have

$$0+2c\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0$$

or

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)=0 \quad \dots(4)$$

Since c is arbitrary constant. The equation (4) is the desired PDEs.

Again, differentiating (2) equation w.r.t. x and (3) w.r.t. y , we get

$$2a+2c\left\{\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right\}=0$$

and

$$2b+2c\left\{\left(\frac{\partial z}{\partial y}\right)^2+z\left(\frac{\partial^2 z}{\partial y^2}\right)\right\}=0$$

Again from (2), $a=-\frac{cz}{x}\frac{\partial z}{\partial x}$. Substituting this in above equation, we obtain

$$-\left(\frac{cz}{x}\right)\times\left(\frac{\partial z}{\partial x}\right)+c\left\{\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right\}=0$$

or $$zx\left(\frac{\partial^2 z}{\partial x^2}\right) + x\left(\frac{\partial z}{\partial x}\right)^2 - z\left(\frac{\partial z}{\partial x}\right) = 0 \quad ...(5)$$

Similarly from (3), we get

$$zy\left(\frac{\partial^2 z}{\partial y^2}\right) + y\left(\frac{\partial z}{\partial y}\right)^2 - z\left(\frac{\partial z}{\partial y}\right) = 0 \quad ...(6)$$

Hence required the PDEs.

<u>RULE2:</u> Derivation of a partial differential equation by elimination of Arbitrary function $\boldsymbol{\phi}$ from the equation $\boldsymbol{\phi(u,v) = 0}$,where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the functions of $\boldsymbol{x, y}$ and $\boldsymbol{z}$.

Let $$\phi(u,v) = 0 \quad ...(1)$$

be the given equation and let

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial y}{\partial x} = 0 \text{ and } \frac{\partial x}{\partial y} = 0 \quad ...(2)$$

Now differentiating (1) w.r.t.x, we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

Now from (2)

$$\frac{\left(\frac{\partial \phi}{\partial u}\right)}{\left(\frac{\partial \phi}{\partial v}\right)} = -\frac{\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)} \quad ...(3)$$

Similarly Differentiating (1) w.r.t. y, we have

$$\frac{\left(\frac{\partial \phi}{\partial u}\right)}{\left(\frac{\partial \phi}{\partial v}\right)} = -\frac{\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)} \quad ...(4)$$

Now eliminating ϕ with the help of (3) and (4), we obtain

$$\frac{\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)} = \frac{\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)}$$

$$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + p\frac{\partial v}{\partial z}\right)$$

or $$Pp + Qq = R$$

where $P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$, $R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

SOLVED EXAMPLE

EXAMPLE1: Solve the partial differential equation by eliminating the arbitrary function f from the equation $x + y + z = f(x^2 + y^2 + z^2)$.

SOLUTION: The given equation is

$$x + y + z = f(x^2 + y^2 + z^2) \qquad ...(1)$$

Differentiating (1) with w.r.t. x and y

$$1 + p = f'(x^2 + y^2 + z^2)\,(2x + 2zp)\ ...(2)$$

$$1 + q = f'(x^2 + y^2 + z^2)\,(2y + 2zq)\ ...(3)$$

From (2) and (3), we obtain

$$\frac{1 + p}{(2x + 2zp)} = \frac{1 + q}{(2y + 2zq)}$$

$$(1 + p)(y + zq) = (1 + q)(x + zp)$$

$$(y + zq) + p(y + zq) = (x + zp) + q(x + zp)$$

$$(y + zq) + py + zqp = (x + zp) + qx + zpq$$

$$(y + zq) + py = (x + zp) + qx$$

$$y + zq + py = x + zp + qx$$

$$zq + py - zp - qx = x - y$$

$p(y - z) + q(z - x) = x - y$ is required the PDEs.

EXAMPLE2: Eliminate the arbitrary functions f and F from $y = f(x - at) + F(x + at)$.

SOLUTION: The given equation is

$$y = f(x - at) + F(x + at) \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{\partial y}{\partial x} = f'(x - at) + F'(x + at)$$

Again, differentiating,

$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) + F''(x + at)\ ...(2)$$

Also, differentiating (1) w.r.t. t, we obtain

$$\frac{\partial y}{\partial t} = f'(x - at)(-a) + F'(x + at)(a)$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x - at)(-a)^2 + F''(x + at)(a)^2$$

$$\frac{\partial^2 y}{\partial t^2} = a^2[f''(x - at) + F''(x + at)]\ ...(3)$$

From (1) and (2), we obtain

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

EXAMPLE3: From the partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$.

SOLUTION: Let the given equation

$$z = f(x^2 - y) + g(x^2 + y) \quad \ldots (1)$$

Now differentiating (1) w.r.t. x and y , we have

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2gx(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}$$

$$\frac{\partial z}{\partial y} = -f'\ (x^2 - y) + g'(x^2 + y)$$

Again differentiating above equation w.r.t. . x and y , we obtain

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'\ (x^2 - y) + g'\ (x^2 + y)\}$$
$$+ 4x^2\{(x^2 - y) + g''\ (x^2 - y)\} \ldots (2)$$
$$\frac{\partial^2 z}{\partial y^2} = f''\ (x^2 - y) + g''\ (x^2 + y)$$

Again (2),

$$f'\ (x^2 - y) + g'(x^2 + y) = \left(\frac{1}{2x}\right) \times \left(\frac{\partial z}{\partial x}\right)$$

Putting the value of $f''\ (x^2 - y) + g''\ (x^2 + y)$ and $f'\ (x^2 - y) + g'(x^2 + y)$ in (2), we get

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right) \times \left(\frac{\partial z}{\partial x}\right) + 4x^2 \frac{\partial^2 z}{\partial y^2}$$

$x \frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial z}{\partial x}\right) + 4x^3 \frac{\partial^2 z}{\partial y^2}$ is required the solution.

1.9 *CAUCHY'S PROBLEM FOR FIRST ORDER PDEs:-*

If

a. $x_0(\mu), y_0(\mu)$ and$z_0(\mu)$ are functions which together with their first derivatives, are continuous in interval I defined by $\mu_1 < \mu < \mu_2$.
b. And if $f(x, y, z, p, q)$ is continuous function of x, y, z, p and q in certain region U of the $xyzpq$ space, then it is required to establish the existence of function $\phi(x, y)$ with the following properties:
   i. $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

ii. For all value of x and y lying in R, the point $\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\}$ lies in U and $f\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\} = 0$.

iii. For all μ belonging to interval I, the point $\{x_0(\mu), y_0(\mu)\}$belongs to the region R and $\phi\{x_0(\mu), y_0(\mu)\} = z_0$.

Stated geometrically, what we wish to prove is that there exists a surface $z = \phi(x,y)$ which passes through the curve C whose parametric equations are given by $x = x_0(\mu), y = y_0(\mu), z = z_0(\mu)$ and every point of which the direction $(p, q, -1)$ of the normal is such that $f(x, y, z, p, q) = 0$.

EXAMPLE: Solve the Cauchy's problem for $zp + q = 1$, when the initial data curve in $x_0 = \mu, y_0 = \mu, z_0 = \frac{\mu}{2}, 0 \leq \mu \leq 1$.

SOLUTION: The given equation

$$f(x, y, z, p, q) = zp + q - 1 = 0 \quad \dots (1)$$

And the given initial data curve

$$x_0 = \mu, y_0 = \mu, z_0 = \frac{\mu}{2}, 0 \leq \mu \leq 1 \quad \dots (2)$$

Now from (1), we have

$\frac{\partial f}{\partial p} = z,$ $\frac{\partial f}{\partial q} = 1$ and $\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu,$ for $0 \leq \mu \leq 1$.

Now we have $\frac{dx}{dt} = \frac{\partial f}{\partial p}$ $\frac{dy}{dt} = \frac{\partial f}{\partial q}$ and $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

And $\frac{dx}{dt} = z$, $\frac{dy}{dt} = 1 \quad \dots (3)$

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right) = pz + q = 1, by\ (1) \quad \dots (4)$$

Now integrating (3) and (4) , we have

$$y = t + C_1, \qquad z = t + C_2 \quad \dots (5)$$

Again from (2) at $t = 0, x(\mu, 0) = \mu, y(\mu, 0) = \mu$ and $z(\mu, 0) = \frac{\mu}{2}$ ... (6)

Using (6), (5) reduce to $y = t + \mu$ and $z = t + \frac{\mu}{2} \dots (7)$

Again from (3) and (7), we get

$$\frac{dx}{dt} = t + \frac{\mu}{2} \text{ so that } x = \frac{1}{2} \times t^2 + \frac{1}{2} \times \mu t + C_3$$

Now Using (6),in above equation, we get

$$x = \frac{1}{2} \times t^2 + \frac{1}{2} \times \mu t + \mu$$

And then solving $y = t + \mu$ with above equation for μ and t in terms of x and y, we obtain

$t = \frac{y-x}{1-\left(\frac{y}{2}\right)}$ and $\mu = \frac{x-(y^2/2)}{1-\left(\frac{y}{2}\right)}$

Substituting these values in $z = t + \frac{\mu}{2}$, the required surface passing through the initial data curve is

$$z = \frac{\left\{2(y-x) + x - \frac{y^2}{2}\right\}}{2-y}.$$

SELF CHECK QUESTIONS

Choose the Correct Option:

1. The equation $ptany + qtanx = sec^2 z$ is of order
   a. 1
   b. 2
   c. 3
   d. 4

2. The equation $\frac{\partial^2 z}{\partial x^2} - 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial y}\right)^2 = 0$ is of order
   a. 1
   b. 2
   c. 3
   d. None

3. The equation $(2x+3y)p + 4xq - 8pq = x + y$ is
   a. Linear
   b. Non-linear
   c. Quasi- linear
   d. Semi-linear

4. The equation $(x+y-z)\left(\frac{\partial z}{\partial x}\right) + (3x+2y)\left(\frac{\partial z}{\partial y}\right) + 2z = x + y$ is
   a. Linear
   b. Quasi-linear
   c. Semi-linear
   d. Non-linear

5. If the coefficient of highest derivative does not contain either the dependent variable or its derivatives such partial differential equation is
   a. Linear
   b. Non-linear
   c. Quasi-linear
   d. Semi-linear

6. Choose the correct option:

a. Every semi-linear partial differential equation is quasi-linear.
b. Every quasi-linear partial differential equation is semi-linear.
c. Every semi-linear partial differential equation is linear
d. Every quasi-linear partial differential equation is linear.

7. A semi-linear partial differential equation which is linear is dependent variable and its derivative, then it is
a. Linear
b. Non-linear
c. Quasi-linear
d. Semi-linear

8. Consider the surfaces $z = F(x, y, a, b)$ then corresponding partial differential equation is of the form
a. $f(x, y, z, p, q) = 0$
b. $f(x, y, p, q) = 0$
c. $f(x, y, z) = 0$
d. $f(p, q) = 0$

9. If we eliminate arbitrary constants from the surface Consider the surfaces $z = F(x, y, a, b)$ then corresponding partial differential equation is of the form $F(x, y, z, p, q) = 0,\ \ a, b$ are constants, then the obtained partial differential equation is
a. Quasi-linear
b. Non-linear
c. Both a and b
d. None

10. Consider the surface $F(u, v) = 0$ where u and v are known functions of x, y, z. After eliminating the arbitrary functions from given surface, we obtain
a. A quasi-linear partial differential equation
b. A semi-linear partial differential equation
c. A non-linear partial differential equation
d. A linear partial differential equation

11. A partial differential equation $z = pq$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ is formed by eliminating arbitrary constants a and b from the equation
a. $z = (a + x) + (a + y)$
b. $z = (a + x)(a + y)$
c. $z = ax + by$
d. $2z = (ax + y)^2 + b$

1.10 *SUMMARY :-*

In this unit we have studied the PDEs, order and degree of PDEs, linear and non-linear PDEs, classification of first order PDEs, origin of PDEs, Cauchy problem for first order PDEs. The partial differential equations continue to be fundamental in various scientific and engineering disciplines. They play a crucial role in fields such as physics, engineering, economics, and biology, providing a powerful mathematical framework for understanding and predicting complex phenomena with multiple variables.

1.11 *GLOSSARY:-*

- **Differential Equation:** An equation that relates one or more functions and their derivatives. In the context of partial differential equations, these equations involve partial derivatives with respect to multiple independent variables.
- **Partial Differential Equation (PDE):** A type of differential equation that involves partial derivatives. It describes a relation between a function and its partial derivatives with respect to two or more independent variables.
- **Cauchy problem for a first-order partial differential equation :** The Cauchy problem for a first-order partial differential equation (PDE) involves specifying initial conditions for the unknown function and its partial derivatives.

1.12 *REFERENCES:-*

- Walter A. Strauss(2008), Partial Differential Equations: An Introduction.
- Stanley J. Farlow (1993), Partial Differential Equations for Scientists and Engineers.
- Lawrence C. Evans(1998), Partial Differential Equations. J.
- David Logan (2015),Applied Partial Differential Equations.

1.13 *SUGGESTED READING:-*

- Sandro Salsa(2008), Partial Differential Equations in Action: From Modelling to Theory.
- Robert C. McOwen (2009), Partial Differential Equations: Methods and Applications.

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations

1.14 *TERMINAL QUESTIONS:-*

(TQ-1): Form partial differential equations by eliminating arbitrary constants a and b from the following relations:

a. $z = a(x + y) + b$
b. $z = ax + by + ab$
c. $z = ax + a^2y^2 + b$
d. $z = (x + a)(x + b)$

(TQ-2): Find the partial differential equation of planes having equal x and y intercepts.

(TQ-3): Find the partial differential equation of all spheres whose centres lie on z −axis.

(TQ-4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding PDEs

a. $z = Ae^{pt} sinpx, (p\ and\ A)$
b. $z = Ae^{-p^2t} cospx, (p\ and\ A)$
c. $z = ax^3 + by^3; (a, b)$
d. $4z = \left[ax + \frac{y}{a} + b\right]^2; (a, b)$
e. $z = ax^2 + bxy + cy^2; (a, b, c)$
f. $z^2 = ax^3 + ab + by^3; (a, b, c)$
g. $ax^2 + z^2 + cy^2 = 1$

(TQ-5) Eliminate arbitrary function f from

a. $z = f(x^2 - y^2)$
b. $z = f(x^2 + y^2)$

1.15 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1.a	2.b	3.b	4.b
5.a	6.b	7. a	8.a
9.b	10.c	11. c	

TERMINAL ANSWERS (TQ'S)

(TQ-1):

a. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

b. $z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

c. $\left(\frac{\partial z}{\partial y}\right) = 2y\left(\frac{\partial z}{\partial x}\right)^2$

d. $z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

(TQ-2): $p - q = 0$

(TQ-3): $xq - yp = 0$

(TQ-4):

a. $\frac{\partial^2 z}{\partial x^2} = \frac{dz}{dt}$

b. $\frac{\partial^2 z}{\partial x^2} = \frac{dz}{dt}$

c. $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 3z$

d. $z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

e. $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 2z$

f. $9x^2y^2z = 6x^3y^2\left(\frac{\partial z}{\partial x}\right) + 6x^2y^3\left(\frac{\partial z}{\partial y}\right) + 4z\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

g. $z\left[z - x\left(\frac{\partial z}{\partial x}\right) - y\left(\frac{\partial z}{\partial y}\right)\right] = 1$

Unit 2: Linear Partial Differential Equations of Order One

CONTENTS:

2.1 *INTRODUCTION:-*

In this unit, we will study appears to cover a broad range of topics related to first-order linear partial differential equations, Lagrange's method, integral surfaces, orthogonal surfaces, and extensions to multiple independent variables. Each of these topics contributes to a deeper understanding of the geometric and analytical aspects of partial differential equations.

2.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- Study Lagrange's equation and method for solving specific linear first-order PDEs. Understand the steps involved and the conditions under which this method is applicable.
- Learn about integral surfaces associated with solutions to linear first-order PDEs. Understand their geometric interpretation and significance in the context of differential equations.
- Explore the concept of surfaces orthogonal to integral surfaces. Understand the relationship between these surfaces and the geometric interpretation of solutions.
- Develop the ability to provide a geometrical description of solutions to linear first-order PDEs.

The main objectives of this unit, learners gain a comprehensive understanding of linear first-order PDEs and their applications, preparing them for more advanced studies in differential equations and mathematical modeling.

2.3 *LAGRANGE EQUATION:-*

Lagrange equations in the context of partial differential equations (PDEs) typically refer to a specific type of quasi-linear first-order PDE. The Lagrange equation of order one is given by:

$$Pp + Qq = R$$

where P, Q and R are the functions of x, y, z.

Example: $xyz + yzp = zx$ is Lagrange equation.

2.4 *GENERAL SOLUTION OF LAGRANGE EQUATION:-*

Theorem: The general solution of Lagrange equation

$$Pp + Qq = R \qquad \dots (1)$$

is

$$\phi(u, v) = 0 \qquad \dots (2)$$

where ϕ is an arbitrary function and

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ ... (3)

are two independent solutions of

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{... (4)}$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if $\frac{u}{v}$ is not merely constant.

Proof: Now differentiating (2) with respect to 'x' and 'y', we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (5)}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (6)}$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (5) and (6), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial x} + p\left(\frac{\partial v}{\partial z}\right) \\ \frac{\partial u}{\partial y} + q\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial y} + q\left(\frac{\partial v}{\partial z}\right) \end{vmatrix} = 0$$

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} = 0 \quad \text{... (7)}$$

Similarly

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we have

$$\left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz = 0 \qquad \text{... (8)}$$

$$\left(\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy + \left(\frac{\partial v}{\partial z}\right)dz = 0 \qquad \text{... (9)}$$

where u and v are independent functions.

From (8) and (9) for $dx : dy : dz$, gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad \ldots(10)$$

Now from (4) and (10), we have

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P}=\frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q}=\frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R}=k,$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=kQ, \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=kR$$

Putting these values in (7), we obtain

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q+\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}=0$$

$$(kP)p+(kQ)q=kR$$

$$k(Pp+Qq)=kR$$

$Pp+Qq=R$ which is given equation (1).

Therefore if $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ (where c_1,c_2 are constants) are two independent solutions of the system of differential equations $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$, then $\phi(u,v)=0$ is the solution of $Pp+Qq=R$, ϕ being arbitrary function.

2.5 *WORKING RULE: -*

Working rule for solving $Pp+Qq=R$ by Lagrange's method:-

Step1: Substitute the given equation in the standard form of a linear first-order partial differential equation.

$$Pp+Qq=R \qquad \ldots(1)$$

Step2: Write down the Lagrange's auxiliary equations for (1) namely,

$$\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$$

Step3: Solve (2) by using the well known methods. Let $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ be two independent solutions of (2).

Step4: The general solution of (1) is obtained in one of the following three equivalent forms:

$\phi(u, v) = 0$, $u = \phi(v)$ or $v = \phi(u)$, ϕ being arbitrary function.

TYPE1 based on rule I for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

Given the partial differential equation $Pp + Qq = R$, the Lagrange's auxiliary equation is given by:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Now, let's consider two fractions, say, $\frac{dx}{P}$ and $\frac{dy}{Q}$. If one of the variables (x or y) is either absent or cancels out, then we can set up a differential equation and integrate.

For example, if $\frac{dx}{P}$ and $\frac{dy}{Q}$, are given, and let's say y is absent or cancels out, then we have:

$$\frac{dx}{P} = \frac{dz}{R}$$

Now, you can integrate this equation with respect to x and z separately:

$$\frac{1}{P}dx = \frac{1}{R}dz$$

Similarly, you can repeat the procedure with another set of two fractions.

For example, if $\frac{dy}{Q}$ and $\frac{dz}{R}$ are given, and z is absent or cancels out, then we have:

$$\frac{dy}{Q} = \frac{dx}{P}$$

Integrate this equation with respect to y and x separately:

$$\frac{1}{Q}dy = \frac{1}{P}dx$$

These integrations will give you solutions involving the variables x, y and z. The constants of integration can be determined by any initial or boundary conditions provided.

SOLVED EXAMPLES

EXAMPLE1: Solve the partial differential equation $2p + 3q = 1$ by Lagrange's methods.

SOLUTION: Let the given Differential equation is

$$Pp + Qq = R \qquad \ldots (1)$$

where $P = 2, Q = 3, R = 1$

The Lagrange‘s auxiliary equations for (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{or} \qquad \frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \qquad ..(2)$$

Taking two fraction of two, we obtain

$$\frac{dx}{2} = \frac{dy}{3} \qquad \text{or} \qquad 3dx - 2dy = 0 \quad ...(3)$$

Now integrating (3), we have

$$3x - 2y = c_1, \quad c_1 \text{ being an arbitrary constant}$$

$\therefore u(x, y, z) = 3x - 2y = c_1$ is one solution of the given partial differential equation

Similarly, taking last two fraction of two, we get

$$\frac{dy}{3} = \frac{dz}{1} \qquad \text{or} \qquad dy - 3dz = 0 \quad ...(4)$$

Now integrating (4), we get

$$y - 3z = c_2, \quad c_2 \text{ being an arbitrary constant}$$

$\therefore v(x, y, z) = y - 3z = c_2$ is another solution of the given partial differential equation.

Hence the general solution is given below

$$\phi = (3x - 2y, y - 3z) = 0$$

Where ϕ is an arbitrary constant.

EXAMPLE2: Find the general solution of $zp + x = 0$.

SOLUTION: Let the given Differential equation is $zp + x = 0$

$$Pp + Qq = R \qquad ...(1)$$

where $P = z, Q = 0, R = -x$

The Lagrange‘s auxiliary equations for (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{or} \qquad \frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \qquad ..(2)$$

Taking first and last fraction of (2), we obtain

$$\frac{dx}{z} = \frac{dz}{-x} \qquad \text{or} \qquad xdx + zdz = 0 \quad ...(3)$$

Now integrating (3), we have

$$\frac{x^2}{2} + \frac{z^2}{2} = k, \quad \text{or} \quad x^2 + z^2 = c_1 \text{ being an arbitrary}$$

constant

$\therefore u(x, y, z) = x^2 + z^2 = c_1$ is one solution of the given partial differential equation.

Also the second fraction of (2), we get

$$dy = 0$$

Integrating, $y = c_2$

$\therefore v(x, y, z) = y = c_2$ is another solution of the given partial differential equation.

Hence the desired solution is

$$\phi = (x^2 + z^2, y) = 0$$

where ϕ is an arbitrary constant.

EXAMPLE3: Solve $y^2p - xyp = -x(z - 2y)$.

SOLUTION: Let the given equation

$$y^2p - xyp = -x(z - 2y) \qquad \dots (1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \qquad \dots (2)$$

Taking the first two fractions of (1) , we get

$2xdx + 2ydy = 0$ so $x^2 + y^2 = c_1$

Now again taking the last two fractions of (1), we have

$\frac{dz}{dy} = \frac{z-2y}{y}$ or $\frac{dz}{dy} + \frac{z}{y} = 2$

So which is linear in z and y. Its integrating factor = $e^{\int (1/y)dy} = e^{\log y} = y$.

Hence $z.y = \int 2ydy + c_2$ or $zy - y^2 = c_2$

From above equations, the required general integral is $\phi(x^2 + y^2, zy - y^2)$ being an arbitrary function.

EXAMPLE4: Solve $p\, tanx + qtany = tanz$.

SOLUTION: The given equation is

$$p\, tanx + qtany = tanz \quad \dots (1)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$$

Taking first two fraction of above equation, we obtain

$$\frac{dx}{tanx} = \frac{dy}{tany} = cotxdx - cotydy = 0$$

Now integrating, $logsinx - logsiny = logc_1$ or $\frac{sinx}{siny} = c_1$

Again last two fraction of above equation, we get

$$\frac{dy}{tany} = \frac{dz}{tanz} = cotydx - cotzdz = 0$$

Now integrating, $logsiny - logsinz = logc_1$ or $\frac{siny}{sinz} = c_2$

Hence the required general solution is $\frac{sinx}{siny} = \phi\left(\frac{siny}{sinz}\right)$, ϕ being an arbitrary function.

TYPE2 based on rule II for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

Let the Lagrange's auxiliary equations for the partial differential

$$Pp + Qq = R \qquad \ldots (1)$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots (2)$$

Suppose that one integral of (2) is known by using rule 1 derived in previous article and suppose also that another integral cannot be derived by using the rule I of previous article. Then, one (the first) integral known to us is used to find another (the second) integral as shown in the following solved examples. Note that in the second integral, the constant of integration of the first integral should be removed later on equation.

SOLVED EXAMPLE

EXAMPLE 1: Solve $xzp + yzq = xy$

SOLUTION: Given $xzp + yzq = xy \qquad \ldots (1)$

From (1), we have

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots (2)$$

Taking first two fraction, we get

$$\frac{dx}{x} - \frac{dy}{y} = 0 \qquad \ldots (3)$$

Integrating (3), we obtain

$\log x - \log y = \log c_1$ or $\frac{x}{y} = c_1$ or $x = yc_1 \quad \ldots (4)$

From second and third fraction of (2), we have

$\frac{dy}{yz} = \frac{dz}{c_1 y^2}$ or $c_1 y dy - z dz = 0 \quad \ldots (5)$

Integrating (3), we have

$$\frac{1}{2}c_1 y^2 - \frac{1}{2}z^2 = c_2$$

$$xy - z^2 = c_2$$

From (4) and (5), the required general solution is $\phi\left(xy - z^2, \frac{x}{y}\right)$, ϕ being an arbitrary function.

EXAMPLE 2: Solve $p + 3q = 5z + \tan\,(y - 3x)$

SOLUTION: Given $p + 3q = 5z + \tan(y - 3x) \qquad \ldots (1)$

From (1), we get

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)} \qquad \ldots (2)$$

Taking first two fraction

$$dy - 3dx = 0$$

Now integrating above equation $y - 3x = c_1$, c_1 being an arbitrary constant.

Again from (2), we obtain

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y - 3x)}$$

Integrating, $x - \frac{1}{5}\log(5z + tanc_1) = \frac{1}{5}\, c_2$

where c_2 being arbitrary constant.

$$5x - \log[5z + \tan(y - 3x)] = c_2$$

Hence the required general integral is

$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$, where ϕ is an arbitrary function.

TYPE3 based on rule III for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

The Lagrange's auxiliary equations for the partial differential

$$Pp + Qq = R \qquad \dots (1)$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots (2)$$

If P_1, Q_1 and R_1 be the function of x, y and z ,then by a well- known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}. \qquad \dots (3)$$

If denominator is zero $(P_1 P + Q_1 Q + R_1 R)$, then $P_1 dx + Q_1 dy + R_1 dz$ is also zero which is integrated to obtain $u_1(x, y, z) = c_1$**.** This method may be repeated to another integral$u_2(x, y, z) = c_2$. Here, $P1$, $Q1$, and $R1$ are called as **Lagrange's multipliers**. As special case, these can be constants also. In such cases second integral should be obtained by using rule I and rule II as the case may be.

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Solve$(mz - ny)p + (nx - lz)q = ly - mx$.

SOLUTION: Given $(mz - ny)p + (nx - lz)q = ly - mx$ ... (1)

The Lagrange's auxiliary equation of (1) is

$$\frac{dx}{(mz - ny)} = \frac{dy}{(nx - lz)} = \frac{dz}{ly - mx}$$

Changing x, y, z multipliers, each fraction of (1), we get

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + n(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ so that $2xdx + 2ydy + 2zdz = 0$

Now integrating, we have

$$x^2 + y^2 + z^2 = c_1 \quad \dots(2)$$

where c_1 being an arbitrary constant.

Again, choose l, m, n multipliers, each fraction of (1), we obtain

$$\frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

$\therefore\ ldx + mdy + ndz = 0$ so that $lx + my + nz = c_2 \quad \dots(3)$

From (2) and (3), the required general solution is given by

$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0,\ \ \phi$ being an arbitrary function.

EXAMPLE2: Solve $x(y^2 - z^2)q - y(z^2 + x^2)q + z(x^2 + y^2)$.

SOLUTION: Given $x(y^2 - z^2)q - y(z^2 + x^2)q + z(x^2 + y^2) \quad \dots(1)$

The Lagrange's auxiliary equation of (1) is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} \quad \dots(2)$$

Changing x, y, z multipliers, each fraction of (2), we have

$$\frac{xdx + ydy + zdz}{x(y^2 - z^2)q - y(z^2 + x^2)q + z(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1 \quad \dots(3)$

Again, choose $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ multipliers, each fraction of (2), we obtain

$$\frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{ldx + mdy + ndz}{0}$$

$\left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\log\{x/(yz)\} = \log c_2$ or $\frac{x}{yz} = c_2 \quad \dots(4)$

$\therefore$ The required solution is $\phi(x^2 + y^2 + z^2, x/(yz)) = 0\ \phi$ being an arbitrary function.

EXAMPLE3: Solve the general solution of the equation$(y + zx)p - (x + yz)q + y^2 - x^2 = 0$.

SOLUTION: Given $(y + zx)p - (x + yz)q + y^2 - x^2 = 0 \quad \dots(1)$

The Lagrange's auxiliary equations are

$$\frac{dx}{(y+zx)} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \quad \dots(2)$$

Changing x, y, z multipliers, each fraction of (2), we get

$$\frac{xdx + ydy - zdz}{x(y + zx) - y(x + yz) - z(x^2 - y^2)} = \frac{xdx + ydy - zdz}{0}$$

$xdx + ydy - zdz = 0$ so that $2xdx + 2ydy - 2zdz = 0$

Integrating, $x^2 + y^2 - z^2 = c_1 \quad \dots(3)$

where c_1 being an arbitrary constant.

Choose $x, y, 1$ multipliers, each fraction of (2), we obtain

$$\frac{xdx + ydy + dz}{y(y+zx) - x(x+yz) + x^2 - y^2} = \frac{xdx + ydy + dz}{0}$$

$\therefore \quad xdx + ydy + dz = 0 \quad$ or $\quad d(xy) + dz = 0$

Integrating, $\quad xy + z = c_2 \quad \ldots (4)$

where c_2 being arbitrary constant.

$\therefore$ Hence the required solution is $\phi\,(x^2 + y^2 - z^2, xy + z) = 0,\ \phi$ being an arbitrary function.

EXAMPLE4: Solve $(y-z)p + (z-x)q = x - y$ **.**

SOLUTION: The Lagrange's auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \quad \ldots (1)$$

Changing 1,1,1 multipliers, each fraction of (1), we obtain

$$\frac{dx + dy + dz}{(y-z) + (z-x) + (x-y)} = \frac{dx + dy + dz}{0}$$

$\therefore \quad dx + dy + dz = 0 \quad \Rightarrow \quad x + y + z = c_1$

Choosing x, y, z multipliers, each fraction of (1), we obtain

$$= \frac{xdx + ydy + zdz}{x(y-z) + y(z-x) + z(x-y)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad 2xdx + 2ydy + 2zdz = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 = c_2$

Hence the required solution is $\phi\,(x + y + z, x^2 + y^2 + z^2) = 0,\ \phi$ being an arbitrary function.

TYPE4 based on rule IV for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

The Lagrange's auxiliary equations for the partial differential

$$Pp + Qq = R \quad \ldots (1)$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots (2)$$

If P_1, Q_1 and R_1 be the function of x, y and z,then by a well- known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R}. \quad \ldots (3)$$

Let us consider that the numerator of (3) is an exact differential of the denominator of (3), then (3) can be combined with a suitable fraction in (2) to obtain an integral. But, in some problems, another set of multipliers P_2, Q_2 and R_2 are so obtain that the fraction

$$\frac{P_2dx + Q_2dy + R_2dz}{P_2P + Q_2Q + R_2R}. \quad \ldots (3)$$

is such that its numerator is an exact differential of denominator. Fractions (3) and (4) are then combined to give an integral. This method may be repeated in some problems to get another integral. Sometimes, only one integral is possible by using the rule IV. In such cases, the second integral should be derived by using rule 1 or rule 2 or rule 3 of previous articles. The following solved examples will illustrate the rule:

SOLVED EXAMPLE

EXAMPLE1: Solve$(y+z)p+(z+x)q=x+y$.

SOLUTION: Given $(y+z)p+(z+x)q=x+y$ ... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad \ldots (2)$$

Changing $1,-1,0$ multipliers, each fraction of (2), we obtain

$$\frac{dx-dy}{(y+z)-(x+y)}=\frac{d(x-y)}{-(x-y)} \quad \ldots (3)$$

Again choose $0,1,-1$ multipliers, each fraction of (2), we get

$$\frac{dy-dz}{(z+x)-(x+y)}=\frac{d(y-z)}{-(y-z)} \quad \ldots (4)$$

Again finally choose 1,1,1 multipliers, each fraction of (2), we have

$$\frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)}=\frac{dx+dy+dz}{2(x+y+z)} \quad \ldots (5)$$

Now from (3),(4) and (5), we have

$$\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)}=\frac{dx+dy+dz}{2(x+y+z)} \quad \ldots (6)$$

Taking first two fraction of (6), we get

$$\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)}$$

∴ Integrating it, we obtain $\log(x-y)=\log(y-z)+\log c_1$

$$\frac{x-y}{y-z}=c_1 \Rightarrow x-y=c_1(y-z) \quad \ldots (7)$$

Taking first and third fraction of (6), we have

$$\frac{2d(x-y)}{(x-y)}+\frac{dx+dy+dz}{2(x+y+z)}=0$$

∴ Integrating it, we get

$$2\log(x-y)+\log(x+y+z)=\log c_2 \Rightarrow (x-y)^2(x+y+z)=c_2$$

$$\Rightarrow (x-y)^2(x+y+z)=c_2 \quad \ldots (8)$$

From (8) and (9), we get

$$\phi\left[\left(\frac{x-y}{y-z},\right)(x-y)^2(x+y+z)\right]=0$$

Where ϕ is an arbitrary function.

EXAMPLE2: Solve $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$.

SOLUTION: Given $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$ ... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)} \qquad \dots(2)$$

Taking first two fraction of (2), we have

$$\frac{dx}{y^2} = \frac{dy}{-x^2}$$

$$x^2dx + y^2dy = 0$$

$\therefore$ Integrating it, we obtain $\quad x^2 + y^2 = c_1 \quad \dots(3)$

choose $1, -1, 0$ multipliers, each fraction of (2), we have

$$= \frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dx - dy}{(y^2+x^2)(x-y)} \quad \dots(4)$$

Combining the third fraction of (2) with (4), we obtain

$$\frac{dx - dy}{(y^2+x^2)(x-y)} = \frac{dz}{z(x^2+y^2)} \Rightarrow \frac{d(x-y)}{(x-y)} - \frac{dz}{z} = 0$$

Integrating it, we get $\log(x-y) - \log(z) = \log c_2$

$$\frac{\log(x-y)}{z} = \log c_2 \Rightarrow \frac{x-y}{z} = c_2$$

Hence, the required solution is

$$\phi\left[x^2 + y^2, \frac{x-y}{z}\right] = 0$$

Where ϕ is an arbitrary function.

2.6 *SURFACES AND NORMALS IN THREE DIMENSION: -*

Suppose Ω be a domain in three-dimensional space R3 and $\phi(x, y, z)$ be a scalar point function, then the vector valued function gradϕ may be obtained as

$$\text{grad}\phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \quad \dots(1)$$

If we suppose that the partial derivatives of ϕ do not vanish simultaneously at any point, then the set of points $\phi(x, y, z)$ in Ω, satisfying the equation

$$\phi(x, y, z) = C \qquad \dots(2)$$

is a surface in Ω for some constant C. This surface is known as a level or **equipotential surface** of ϕ. If (x_0, y_0, z_0) is a obtained point in Ω, then by taking $\phi(x_0, y_0, z_0) = C$, we get an equation of the form

$$\phi(x, y, z) = \phi(x_0, y_0, z_0) \qquad \dots(3)$$

which represents a surface in the domain Ω of three dimensional space passing through the point (x_0, y_0, z_0). Here, equation (2) represents a one-parameter family of surface in the domain Ω. The value of grad ϕ is a vector, normal to the level surface. Now, one may ask, if it is possible to solve equation (2) for z in terms of x and y. To answer this question, suppose a set of relations of the form

$x = f_1(u, v),\ y = f_2(u, v),\ z = f_3(u, v)$ ... (4)

Here, for every pair of values of u and v, we will have three numbers x, y and z, which represent a point in space. But if the Jacobian

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} \neq 0 \qquad \text{... (5)}$$

then, the first two equations of (4) can be solved and u and v can be indicated as functions of x and y such as

$$u = \lambda(x, y) \quad \& \quad v = \mu(x, y) \qquad \text{.... (6)}$$

Hence, the third relation of equation (4) gives the value of z in the form

$$z = f_3[\lambda(x, y), \mu(x, y)\,] \qquad \text{... (7)}$$

This relation is of course, a functional relation between the co-ordinates x, y and z as in equation (2). Hence, any points (x, y, z) obtained from equation (4) always lie on a fixed surface. The set of equations (4) are called as the **parametric equations of a surface**. It may be noted that the parametric equations of a surface need not be unique, which can be seen in the following example:

Let the two sets of parametric equations are

$x = r sin\theta cos\phi \qquad y = r cos\theta sin\phi \qquad z = r cos\theta \quad ...(set 1)$

and

$x = r\frac{(1-\phi^2)}{(1+\phi^2)}cos\theta \qquad y = r\frac{(1-\phi^2)}{(1+\phi^2)}sin\theta \qquad z = r\frac{2r\phi}{1+\phi^2} \quad ...(set 2)$

represent the surface $x^2 + y^2 + z^2 = r^2$, which is a sphere

Now let we take the surface whose equation is

$$z = f(x, y) \qquad \text{... (9)}$$

The above equation may be expressed as

$$\phi = f(x, y) - z = 0 \qquad \text{... (10)}$$

So differentiating with respect to x and y, we have

$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y} = 0$

Using form (9), we obtain

$\frac{\partial z}{\partial x} = \frac{\partial\phi/\partial x}{\partial\phi/\partial z} = \frac{\partial\phi}{\partial x}$ i.e. $\frac{\partial\phi}{\partial x} = p,\ \frac{\partial\phi}{\partial y} = q, \frac{\partial\phi}{\partial z} = -1$ ... (11)

Hence, the direction cosines of the normal to the surface at point $P(x, y, z)$ are obtained by

$$\left(\frac{p}{\sqrt{p^2 + q^2 + 1}}, \frac{q}{\sqrt{p^2 + q^2 + 1}}, \frac{-1}{\sqrt{p^2 + q^2 + 1}}\right) \qquad \text{... (12)}$$

Now, replacing to the level surface obtained by equation (2), it is easy to write the equation of the tangent plane to the level surface at a point (x_0, y_0, z_0) as

$$(x - x_0)\left[\frac{\partial F}{\partial x}\right]_{(x_0,y_0,z_0)} + (y - y_0)\left[\frac{\partial F}{\partial y}\right]_{(x_0,y_0,z_0)} + (x - z_0)\left[\frac{\partial F}{\partial z}\right]_{(x_0,y_0,z_0)} = 0$$

2.7 *CURVE IN THREE DIMENSIONS: INTERECTION OF TWO SURFACES:-*

A curve in three-dimensional space R^3 can be explained in terms of parametric equations. Suppose r denotes the position vector of a point on a curve C, then the vector equation of the curve C may be given as

$$\vec{r} = \vec{F}t, \qquad t \in I \qquad \dots (1)$$

where I is some interval on the real axis. The equation (1) can be written as

$$x = f_1(t), \qquad y = f_2(t), \qquad z = f_3(t) \quad \dots (2)$$

where$\vec{r} = (x, y, z)$, $\vec{F} = [\, f_1(t),\ f_2(t), f_3(t)]$

Now we assume that

$$\left(\frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \frac{df_3(t)}{dt}\right) \neq (0,0,0)$$

This non-vanishing vector is known as the tangent vector to the curve C at the point (x, y, z) or at $[\, f_1(t),\ f_2(t), f_3(t)]$ to the curve C. Another way of explaining a curve in three-dimensional space R^3 is by using the fact that the intersection of the surfaces obtain rise to a curve.

Suppose

$$\phi_1(x, y, z) = C_1 \quad \text{and} \quad \phi_2(x, y, z) = C_2 \quad \dots (3)$$

are two surfaces. Their intersection, if not empty, is always a curve, produced grad ϕ_1 and grad ϕ_2 are not collinear at any point of the domain Ω. In other words, the intersection of surfaces obtained by equation (3) is a curve if

$$\text{grad } \phi_1(x, y, z) = C_1.\ \text{grad}\phi_2(x, y, z) = C_2 \neq (0,0,0)$$

for every (x, y, z) $\in \Omega$. For various values of C1 and C2, equation (4) explains different curves. The totality of these curves is called a two parameter family of curves. Here, C1 and C2 are mentioned to as parameters of this family.

2.8 *INTEGRAL SURFACES PASSING THROUGH A GIVEN CURVE (THE CAUCHY PROBLEM):-*

In the previous article, we have expressed general integral of the partial differential equation $Pp + Qq = R$. We shall now present two methods for finding the integral surface which passes through a obtained curve.

MethodI: Suppose $Pp + Qq = R$ ... (1)

be given partial differential equation. Let its Lagrange's auxiliary equations give us the following two independent solutions

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ ... (2)

Let, we desire to give the integral surface which passes through the curve whose equation in parametric form is obtained by

$$x = x(t), y = y(t), z = z(t)$$

Where t is parameter and may be written as

$u\big(x(t), y(t), z(t)\big) = c_1$ and $v\big(x(t), y(t), z(t)\big) = c_2$

We now eliminate the parameter t from the above equations and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and give the required integral surface.

MethodII: Suppose $Pp + Qq = R$ ... (1)

be given partial differential equation. Let its Lagrange's auxiliary equations give us the following two independent solutions

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ ... (2)

Let, we desire to give the integral surface which passes through the curve which is established by the following two equations

$\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$ ... (3)

We now eliminate x, y and z from the two pairs of given above equations and get a relation between c1 and c2. Finally, we choose c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and we give the desired integral surface.

2.9 *SURFACE ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES:-*

Let $$f(x, y, z) = c \qquad \ldots(1)$$

shows a system or surfaces, where c is a parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Let the surface $$z = \phi(x, y) \qquad \ldots(2)$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ $i.e., p, q, -1$. Hence normal's at P(x, y, z) to (1) and (2) are at right angles, therefore, we obtain

$$p\left(\frac{\partial f}{\partial x}\right) + q\left(\frac{\partial f}{\partial y}\right) - \left(\frac{\partial f}{\partial z}\right) = 0 \quad \text{or} \quad p\left(\frac{\partial f}{\partial x}\right) + q\left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial z}\right) \qquad \ldots(3)$$

Which is the form of $Pp + Qq = R,$ where $P = \left(\frac{\partial f}{\partial x}\right), Q = \left(\frac{\partial f}{\partial y}\right)$ and $R = \left(\frac{\partial f}{\partial z}\right)$.

Conversely, we may easily verify that any solution of (3) is orthogonal to every surface of (1).

2.10 *GEOMETRICAL DESCRIPTION OF SOLUTIONS OF LAGRANGE'S EQUATION $Pp + Qq = R$ AND LANGRAGE'S AUXILIARY EQUATIONS $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$:-*

Let

$$Pp + Qq = R \qquad \ldots(1)$$

and

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots (2)$$

where P, Q, R are the function of x, y, z.

Let $\qquad z = \phi(x, y) \qquad \dots (3)$

Represents the solution of the Lagrange's partial differential equation (1). Then (3) expresses a surface whose normal at any point (x, y, z) has direction ratios$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ $i.e., p, q, -1$. Also, we know that the system of simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we obtain

$$Pp + Qq + R(-1) = 0 \qquad \dots (4)$$

which expresses that the normal to the surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence, the member must touch the surface at that point. Since this holds for each point on (3), therefore, we consider that the curves (2) lies completely on the surface (3) whose differential equation is obtain by (1).

2.11 *GEOMETRICAL INTERPRETATION OF* $\boldsymbol{Pp + Qq = R}$ *:-*

To show that the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solution of

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots (1)$$

Are orthogonal to the system of surfaces whose satisfied the equation

$$Pdx + Qdy + Rdz = 0 \qquad \dots (2)$$

Again, from (1) lie on the surface represented by

$$Pp + Qq = R \qquad \dots (3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

2.12 *LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE WITH n INDEPENDENT VARIABLES:-*

Let $x_1, x_2, x_3, \dots \dots \dots . . x_n$ be the n independent variables and z be dependent function depending on $x_1, x_2, x_3, \dots \dots \dots . . x_n$. Also, let $p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, p_3 = \frac{\partial z}{\partial x_3}, \dots \dots \dots \dots . p_n = \frac{\partial z}{\partial x_n}$

Then, the general linear partial differential equation of order one with n independent variables is obtained by

$$P_1p_1 + P_2p_2 + P_3p_3 + \cdots\cdots + P_np_n = R \qquad (1)$$

where $P_1P_2P_3 \dots\dots\dots P_n$are the functions of $x_1, x_2, x_3, \dots\dots\dots.. x_n$ and R is the function of $x_1, x_2, x_3, \dots\dots\dots.. x_n$ and z.

Therefore, the system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \cdots\cdots = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad \dots(2)$$

Let $u_1(x_1, x_2, x_3, \dots\dots\dots.. x_n, z) = c_1, u_2(x_1, x_2, x_3, \dots\dots\dots.. x_n, z) = c_2, u_3(x_1, x_2, x_3, \dots\dots\dots.. x_n, z) = c_3, \dots\dots\dots\dots n(x_1, x_2, x_3, \dots\dots\dots.. x_n, z) = c_n,$) be any independent integral of (2), then the general solution of (2) is written by

$$\phi(u_1, u_{2,} u_3 \dots\dots.. u_n) = 0$$

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Solve $(y+z)p + (z+x)q = x + y$

SOLUTION: Here the Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \dots(1)$$

Changing $1, -1, 0$ as multipliers of (1), we have

$$\frac{dx - dy}{(y+z) - (z+x)} = \frac{d(x-y)}{-(x-y)} \qquad \dots(2)$$

Again, choosing $0, 1, -1$ as multipliers of (1), we obtain

$$\frac{dy - dz}{(z+x) - (x+y)} = \frac{d(x-y)}{-(y-z)} \qquad \dots(3)$$

Finally, choosing 1,1,1 as multipliers of (1), we get

$$= \frac{dx + dy + dz}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \dots(4)$$

From (2), (3) and (4), we get

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x-y)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \dots(5)$$

Taking the two fractions of (5), we have

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x-y)}{-(y-z)}$$

Integrating, $\log(x-y)=\log(y-z)+\log c_1$, where c_1 being an arbitrary constant.

$\log\left(\frac{x-y}{y-z}\right)=\log c_1$ or $\left(\frac{x-y}{y-z}\right)=c_1$

Again taking first and third fraction of (5),

$$2\frac{d(x-y)}{(x-y)}+\frac{d(x+y+z)}{(x+y+z)}=0$$

Integrating, $2\log(x-y)+\log(x+y+z)=\log c_2$ or

$$(x-y)^2(x+y+z)=c_2$$

Hence the required general solution is $\phi\left[(x-y)^2(x+y+z),\frac{x-y}{y-z}\right]=0$, ϕ being arbitrary function.

EXAMPLE2: Solve $(1+y)p+(1+x)q=z$

SOLUTION: Here the Lagrange's auxiliary equations are

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z} \qquad \dots(1)$$

Taking the first two fractions of (1), we get

$(1+x)dx=(1+y)dy$ or $2(1+x)dx-2(1+y)dy=0$ ... (2)

Integrating,$(1+x)^2-(1+y)^2=c_1$, c_1 being an arbitrary constant.

Taking 1,1,0 as multipliers of each fraction of (1)

$$=\frac{dx+dy}{1+y+1+x}=\frac{d(2+x+y)}{2+x+y} \qquad \dots(3)$$

Combining the last fraction of (1) with (3), we have

$\frac{d(2+x+y)}{2+x+y}=\frac{dz}{z}$ or $\frac{d(2+x+y)}{2+x+y}-\frac{dz}{z}=0$

Integrating, $\log(2+y+x)-\log z=\log c_2$ or $\frac{(2+x+y)}{z}=c_2$ ... (4)

From (2) and (4), the required general solution is obtained by

$\phi\left[(1+x)^2-(1+y)^2,\frac{(2+x+y)}{z}\right]=0,\phi$ being an arbitrary function.

EXAMPLE3: Find the tangent vector at the point $\left(0,1,\frac{\pi}{2}\right)$ to the helix described by the parametric equations $x=cost, y=sint, z=t$.

SOLUTION: The tangent vector to the helix at (x,y,z) is obtained by

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=(-sint,cost,1)$$

We state that the given point $\left(0,1,\frac{\pi}{2}\right)$ corresponds $t=\frac{\pi}{2}$. Therefore, the required tangent to vector to helix is obtain by

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=(-sint,cost,1)=(-1,0,1)$$

EXAMPLE4: Find the equation of the tangent line to the space circle $x^2+y^2+z^2=1, x+y+z=0$ at the point $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},-\frac{3}{\sqrt{14}}\right)$.

SOLUTION: Let
$F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ and

$$F(x,y,z) = x + y + z = 0 \qquad \dots(1)$$

The equation of the tangent line at the point (x_0, y_0, z_0) is

$$\frac{(x - x_0)}{\frac{\partial(F,G)}{\partial(y,z)}} = \frac{(y - y_0)}{\frac{\partial(F,G)}{\partial(z,x)}} = \frac{(z - z_{0)}}{\frac{\partial(F,G)}{\partial(x,y)}} \qquad \dots(2)$$

Where

$$\frac{\partial(F,G)}{\partial(y,z)} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y} = 2y - 2z = \frac{4}{\sqrt{14}} + \frac{6}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

$$\frac{\partial(F,G)}{\partial(z,x)} = \frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z} = 2z - 2x = -\frac{6}{\sqrt{14}} - \frac{2}{\sqrt{14}} = \frac{8}{\sqrt{14}}$$

$$\frac{\partial(F,G)}{\partial(x,y)} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} = 2z - 2y = \frac{2}{\sqrt{14}} - \frac{4}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

The required solution of the point $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$ is obtained by

$$\frac{\left(x - \frac{1}{\sqrt{14}}\right)}{\frac{10}{\sqrt{14}}} = \frac{\left(y - \frac{2}{\sqrt{14}}\right)}{\frac{8}{\sqrt{14}}} = \frac{\left(z - \frac{3}{\sqrt{14}}\right)}{\frac{-2}{\sqrt{14}}}$$

EXAMPLE5: Find the integral surface of the linear partial differential equation$x(x^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$.

SOLUTION: Given $x(x^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z \quad \dots(1)$

The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{x(x^2 + z)} = \frac{dy}{y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z} \qquad \dots(2)$$

The two independent solutions of (2) may be given as

$$u(x,y,z) = xyz = c_1 \qquad \dots(3)$$

$$v(x,y,z) = x^2 + y^2 - 2z = c_2 \qquad \dots(4)$$

Taking *t* as parameter, the obtained equation of the straight line $x + y = 0, z = 1$ can be put in parametric form

$$x = t, y = -t, \qquad z = 1 \qquad \dots(5)$$

Putting the value of (5) in (3) and (4), we have $-t^2 = c_1$ and $2t^2 - 2 = c_2 \Rightarrow -2c_1 - 2 = c_2 \;\Rightarrow\; 2c_1 + 2 + c_2 = 0 \qquad \dots(6)$

Now, substituting the values of c_1 and c_2 from (3) and (4) in (6), we obtain

$$2xyz + x^2 + y^2 - 2z = 0$$

which is the desired integral surface of the given PDE.

EXAMPLE6: Find the equation of integral surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2 + z^2 = 1, x + z = 2$.

SOLUTION: Given $4yzp + q + 2y = 0$ ... (1)
The equation of the obtained curve is

$$y^2 + z^2 = 1, \;\; x + z = 2 \qquad \text{... (2)}$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y} \qquad \text{... (3)}$$

Taking the first and third fractions of (3), we obtain
$dz + 2zdz = 0$ so that $x + z^2 = 2$ ... (4)
Taking the last two fractions of (3), we get
$dz + 2ydy = 0$ so that $x + y^2 = 2$ ... (5)
Adding (4) and (5), we have

$$(z^2 + y^2) + (x + z) = c_1 + c_2$$

From (2), $1 + 2 = c_1 + c_2$
Substituting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required integral surface is written by
$3 = x + z^2 + y^2 + z$ or $x + z^2 + y^2 + z - 3 = 0$

EXAMPLE7: Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.
SOLUTION: The equation of the given system of surfaces is

$$f(x, y, z) \equiv \frac{z(x + y)}{3z + 1} = C \qquad \text{... (1)}$$

$$\therefore \; \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \frac{\partial f}{\partial z} = \left[\frac{3z+1-3z}{(3z+1)^2}\right](x + y) = \frac{(x+y)}{(3z+1)^2}$$

$$z(3z + 1)q + z(3z + 1)q = x + y \qquad \text{... (2)}$$

The Lagrange's auxiliary equations is

$$\frac{dx}{z(3z + 1)} = \frac{dy}{z(3z + 1)} = \frac{dz}{(x + y)} \qquad \text{... (3)}$$

Taking first two fraction of (3), we have $dx - dy = 0$
Integrating it, $x - y = c_1$
Taking $x, y, -z(3z + 1)$ as multipliers, each fraction of (3) is

$$xdx + ydy - z(3z + 1)dz = 0$$

$$xdx + ydy - 3z^2dz - zdz = 0$$

Or

$$2xdx + 2ydy - 6z^2dz - 2zdz = 0$$

Integrating it, we obtain $x^2 + y^2 - 2z^3 - z^2 = c_2$ ... (4)

Hence, the equation (1) is given by

$$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y)$$

where ϕ is an arbitrary function.

EXAMPLE8: Find the family orthogonal to $\phi[z(x+y)^2, x^2-y^2]=0$.
SOLUTION: Given

$$\phi[z(x+y)^2, x^2-y^2]=0 \qquad \ldots(1)$$

Let $u=z(x+y)^2$ $\qquad v=x^2-y^2$
From (1) becomes,

$$\phi(u,v)=0 \qquad ..(2)$$

Differentiating two w.r.t. x and y, we have

$$\left.\begin{aligned}\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0\\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+q\frac{\partial v}{\partial z}\right)=0\end{aligned}\right\}\ldots(3)$$

From (2),

$\frac{\partial u}{\partial x}=2z(x+y)$, $\frac{\partial u}{\partial y}=2z(x+y)$, $\frac{\partial u}{\partial z}=(x+y)^2$, $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial y}=-2y$, $\frac{\partial v}{\partial z}=0$ $\qquad \ldots(4)$

Substituting the values of (4) in (3), we get

$$\left(\frac{\partial\phi}{\partial u}\right)[2z(x+y)+p(x+y)^2]+\left(\frac{\partial\phi}{\partial v}\right)[2x+0]=0$$

$$\left(\frac{\partial\phi}{\partial u}\right)[2z(x+y)+q(x+y)^2]+\left(\frac{\partial\phi}{\partial v}\right)[-2y+0]=0$$

Now

$$\frac{\frac{\partial\phi}{\partial u}}{\frac{\partial\phi}{\partial v}}=\frac{2x}{2z(x+y)+p(x+y)^2}=\frac{-2y}{2z(x+y)+p(x+y)^2}$$

$$x(x+y)[2z+p(x+y)]=-y(x+y)[2z+q(x+y)]$$

$$2xz+xq(x+y)+2zy+yp(x+y)=0$$

$$xq(x+y)+py(x+y)=-2z(x+y)$$

$$(qx+py)(x+y)=-2z(x+y)$$

$$qx+py=-2z \qquad \ldots(4)$$

which is a partial differential equation of the family of surfaces given by (1).
The differential equation of the family of surfaces orthogonal to (4) is obtain by

$ydx+xdy-2zdz=0$ or $d(xy)-2zdz=0$
Integrating it,

$$xy-z^2=c$$

which is the represented family of orthogonal surfaces.

EXAMPLE9: Solve $x_2x_3p_2 + x_3x_1p_2 = x_1x_2x_3$

SOLUTION: The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \cdots + P_np_n = R$, we obtain

$P_1 = x_2x_3, P_2 = x_3x_1, P_3 = x_1x_2$ and $R = x_1x_2x_3$

$\therefore$ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \frac{dx_3}{p_3} \quad \text{or} \quad \frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{x_1x_2x_3} \qquad \ldots(1)$$

Taking first and second fraction of (1), we have

$$x_1dx_1 = x_2dx_2$$

so

$$\frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2}$$

which give

$$u_1 = x_1^2 - x_2^2 = C_1 \qquad \ldots(2)$$

Taking second and third fraction of (1), we have

$$x_2dx_2 = x_3dx_3$$

so

$$\frac{x_2^2}{2} = \frac{x_3^2}{2} + \frac{C_2}{2}$$

which give

$$u_2 = x_2^2 - x_3^2 = C_2 \qquad \ldots(3)$$

Taking third and fourth fraction of (1), we have

$$dz = x_3dx_3$$

so

$$z = \frac{x_3^2}{2} + \frac{C_3}{2}$$

which give

$$u_3 = 2z - x_3^2 = C_3 \qquad \ldots(4)$$

Finally, from (2), (3) and (4), the general solution of the obtained partial differential equation is

$$\phi = (x_1^2 - x_2^2, x_2^2 - x_3^2, 2z - x_3^2) = 0.$$

EXAMPLE9: Solve $P_1p_1 + P_2p_2 + P_3p_3 = az + (x_1x_2)/x_3$

SOLUTION: The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \cdots + P_np_n = R$, we obtain

$P_1 = x_1, P_2 = x_2, P_3 = x_3$ and $R = az + (x_1x_2)/x_3$

$\therefore$ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} \quad \text{or} \quad \frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{az+(x_1x_2)/x_3} \qquad \dots (1)$$

Taking first and second fraction of (1), we have

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2}$$

so

$$\log x_1 = \log x_2 + \log c_1$$

which give

$$u_1 = \frac{x_1}{x_2} = C_1 \qquad \dots (2)$$

Taking second and third fraction of (1), we have

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3}$$

so

$$\log x_2 = \log x_3 + \log c_2$$

which give

$$u_1 = \frac{x_2}{x_3} = C_2 \qquad \dots (3)$$

Taking first and fourth fraction of (1), we have

$$\frac{dx_1}{x_1} = \frac{dz}{az+(x_1x_2)/x_3} = \frac{dz}{az+C_2x_1} \qquad \text{since } \frac{x_2}{x_3} = C_2$$

$$\frac{dz}{dx_1} = \frac{az + C_2x_1}{x_1}$$

i.e.,

$$\frac{dz}{dx_1} - \frac{a}{x_1}z = C_2 \quad \dots (4)$$

which is a linear differential equation whose integrating function (I.F.) is given as follows :

so

$I.F\ of\ (4)$, we have

$$= e^{-a\int\frac{dx_1}{x_1}} = e^{-a\log x_1} = x_1^{-a}$$

The solution of the linear differential equation (4) is given by

$$zx_1^{-a} = C_2 \int x_1^{-a} dx + C_3$$

$$zx_1^{-a} = C_2 \frac{x_1^{-a+1}}{1-a} + C_3$$

$$zx_1^{-a} = \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) + C_3 \text{ since } \frac{x_2}{x_3} = C_2$$

$$\frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) = C_3$$

i.e.,

$$u_3 = \frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) = C_3 \quad \ldots (5)$$

Finally, from (2), (3) and (5), the general solution of the given partial differential equation is

$$\phi\left(\frac{x_1}{x_2}, \frac{x_2}{x_3}, \left\{\frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right)\right\}\right) = 0.$$

SELF CHECK QUESTIONS

Choose the Correct Option:

1. The PDE $Pp + Qq = R$ is popularly known as
   a. Lagrange's equation
   b. Euler's equation
   c. Monge's equation
   d. Leibnitz equation
2. Lagrange's auxiliary equations for $xzp + yzq = xy$ are
   a. $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$
   b. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$
   c. $\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{1}$
   d. $\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{z}$
3. The integral surface satisfying $4yzp + q + 2y =$ and passing Through $y^2 + z^2 = 1, x + z = 2$ is
   a. $y^2 + z^2 + x + z - 3 = 0$
   b. $y^2 + z^2 + z + x = 0$
   c. $y^2 + z^2 + z + x - 3 = 0$
   d. $y^2 + z^2 + y + z = 0$
4. The solution of the PDE $xzp + yzq = xy$ is
   a. $\phi\left(\frac{x}{y}, xy - z^2\right) = 0$

b. $\phi(x^2, y) = 0$
c. $\phi(x^2z, y) = 0$
d. $\phi(x^2zx, yx) = 0$

2.13 *SUMMARY*

In this unit we have studied Lagrange's equation, General solution of Lagrange Equation, working rule (Example based), Integral Surface, Surfaces orthogonal, Geometrical description of solutions of $Pp + Qq = R$ and of the system of equations $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$ and to establish relationship and Linear Partial Differential Equations of order one with n independent variables and understanding and solving linear first-order PDEs are fundamental in the study of more complex partial differential equations and their applications in physics, engineering, and other scientific disciplines.

2.14 *GLOSSARY:-*

- **Partial Differential Equation (PDE):** An equation involving partial derivatives of an unknown function with respect to two or more independent variables.
- **Linear PDE:** A PDE where the unknown function and its derivatives appear linearly (i.e., without products or powers) in the equation.
- **Order One:** Refers to the highest order of derivatives present in the PDE. For linear PDEs of order one, the highest derivative is first-order.
- **Dependent Variable:** The variable that depends on other variables. In PDEs, this is typically the function being solved for.
- **Independent Variables:** Variables with respect to which partial derivatives are taken. In PDEs, these represent dimensions or parameters that influence the behavior of the dependent variable.
- **Coefficient Functions:** Functions that multiply the derivatives of the dependent variable in the PDE. These coefficients may depend on the independent variables.
- **Characteristics:** Curves or surfaces along which the PDE simplifies to an ordinary differential equation (ODE). They help in transforming the PDE into a simpler form for solution.
- **Integral Surface:** A surface that satisfies a given PDE. Solutions to linear first-order PDEs often involve finding such surfaces.

- **Initial Conditions:** Specified values or conditions given at a particular point in the domain of the PDE, often used to determine a unique solution.
- **Boundary Conditions:** Conditions specified on the boundary of the domain, essential for determining a unique solution to the PDE.
- **Method of Characteristics:** A technique used to solve linear first-order PDEs. It involves finding characteristic curves along which the PDE reduces to an ODE.
- **Compatibility Conditions:** Conditions that ensure the existence and uniqueness of solutions to the PDE, often related to the coefficients and boundary/initial conditions.
- **Transport Equation:** A specific type of linear first-order PDE that describes the advection or transport of a quantity along characteristic curves.
- **Orthogonal Surfaces:** Surfaces that intersect at right angles. In the context of PDEs, understanding orthogonal surfaces can provide geometric insights into the solutions.
- **General Solution:** The set of all possible solutions to the PDE, often involving arbitrary functions or constants determined by initial/boundary conditions.

This glossary provides a foundational understanding of terms related to linear first-order PDEs. Each term plays a crucial role in formulating, understanding, and solving these equations in various applications across science and engineering.

2.15 *REFERENCES:-*

- Sandro Salsa(2008), Partial Differential Equations in Action: From Modelling to Theory.
- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations

2.16 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- David Logan (2015), Applied Partial Differential Equations.

2.17 *TERMINAL QUESTIONS:-*

(TQ-1): Solve $p + 3q = 5z + \tan(y - 3x)$.

(TQ-2): Solve $z(z^2 + xy)(px - qy) = x^4$.

(TQ-3): Solve $xyp + y^2q = zxy - 2x^2$

(TQ-4): Solve $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

(TQ-5): Solve $xzp + yzq = xy$

(TQ-6): Solve $(y - zx)p + (x + yz)q = x^2 + y^2$

(TQ-7): Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

(TQ-8): Solve $(y - z)p + (z - x)q = x - y$

(TQ-9): Solve $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$

(TQ-10): Solve$2y(z - 3)p + (2x - z)q = y(2x - 3)$

(TQ-11): Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$

(TQ-12): Solve $ptanx + qtany = tanz$

(TQ-13): Solve $zp = -x$

(TQ-14): Find the general solution of differential equation

$$x^2\left(\frac{\partial z}{\partial x}\right) + y^2\left(\frac{\partial z}{\partial y}\right) = (x + y)z$$

(TQ-15): Find the general solution of differential equation

$$px(x + y) - qy(x + y) + (x - y)(2x = 2y + z) = 0$$

(TQ-16): Solve $p + q = x + y + z$

(TQ-17): Find the integral surface of the partial differential equation $(x - y)p + (y - x - z)q = z$ passing through the circle $z = 1, x^2 + y^2 = 1$.

(TQ-18): Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.

(TQ-19): Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = cy^2$

(TQ-20): Find the surface which is orthogonal to the one parameter System$z = cxy(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2z = 0$.

(TQ-21): Find the integral surface of the linear partial differential equation $x(x^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line$x + y = 0, z = 1$.

(TQ-22): Find the equation of the integral surface of the partial differential equation $2y(z-3)p+(2x-z)q=y(2x-3)$ which passes through the circle $z=0,\ \ x^2+y^2=2x$

2.18 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1. (a) 2.(a) 3.(a) 4.(a)

TERMINAL ANSWERS (TQ'S)

(TQ-1): $5x-\log[5z+tan(y-3x)]=\phi(y-3x)$
(TQ-2): $\phi(xy,x^4-z^4-2xyz^2)=0$
(TQ-3): $x-\log[z-2(x^2/y^2)]=\phi(x/y)$
(TQ-4): $(y^2-ax-x^3)/x=\phi\left(\frac{z}{y}\right)$
(TQ-5): $\phi\left(xy-z^2,\frac{x}{y}\right)=0$
(TQ-6): $\phi(x^2-y^2+z^2,xy-z)=0$
(TQ-7): $\phi(x^2+y^2-2z,xyz)=0$
(TQ-8): $\phi(x+y+z,x^2+y^2+z^2)=0$
(TQ-9): $\phi(x^2+y^2-z^2,xy+z)=0$
(TQ-10): $\phi\left(\frac{1}{y}-\frac{1}{x},\frac{xy}{z}\right)=0$
(TQ-11): $\phi(x^3-y^3,x^2-z^2)=0$
(TQ-12): $\frac{sinx}{siny}=\phi\left(\frac{siny}{sinz}\right)$
(TQ-13): $\phi(x^2+y^2,zy-y^2)=0$
(TQ-14): $\phi\left(\frac{x,y}{z},\frac{x-y}{z}\right)=0$
(TQ-15): $\phi[xy,(x+y)(x+y+z)]=0$
(TQ-16): $\phi[x-y,e^{-x}(2+x+y+z)]=0$
(TQ-17): $z^4(x+y+z)^2+(x-y-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0$
(TQ-18): $x^2+y^2-2z^3-z^2=\phi(x-y)$
(TQ-19): $\frac{(x^2+y^2+z^2)}{z}=\phi[(2x^2+y^2)/z^2]$
(TQ-20): $(x^2+y^2+4z^2)^2(x^2-y^2)^2=a^2(x^2+y^2)$
(TQ-21): $2xyz+x^2+y^2-2z+2=0$
(TQ-22): $x^2-y^2-z^2-2x+4z=0$

Unit 3: Non-linear Partial Differential Equations of Order One

CONTENTS:

3.1 *INTRODUCTION:-*

Non-linear partial differential equations (PDEs) of order one are mathematical expressions that involve partial derivatives of a dependent variable with respect to one or more independent variables. The non-linearity arises when these partial derivatives occur in terms other than the first degree, meaning they may appear squared, cubed, or in some other non-linear form. The study of non-linear PDEs is crucial in various scientific disciplines, including physics, engineering, biology, and economics, among others. These equations can describe complex phenomena that involve interactions, feedback mechanisms, or non-trivial dependencies.

Non-linear PDEs of order one are often more challenging to solve analytically compared to linear PDEs. The non-linear terms introduce complexities that may not have closed-form solutions in many cases. As a result, researchers and scientists often resort to numerical methods, such as finite difference, finite element, or spectral methods, to approximate solutions to these equations. The solutions to non-linear PDEs can exhibit interesting behaviors, including the formation of shocks, solutions, and other nonlinear structures. Understanding and solving these equations are crucial for gaining insights into the behavior of physical and natural systems. Researchers employ various techniques to tackle non-linear PDEs, such as perturbation methods, similarity transformations, and numerical simulations. Additionally, the development of computational tools and advancements in numerical algorithms has played a significant role in studying and solving non-linear PDEs in practical applications.

3.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- To complete integral provides a general solution to a differential equation. It represents a family of solutions that includes all possible solutions, with the inclusion of arbitrary constants or functions.
- To understand compatible system of first order equations.
- To provide solution of Charpit's Method.

3.3 *COMPLETE INTEGRAL OR COMPLETE SOLUTION:-*

Let us consider a relation

$$\phi(x, y, z, a, b) = 0 \qquad \dots(1)$$

in the variables x, y and z, where x, y are independent variables, z is a dependent variable and a, b are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we obtain

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\phi z} p = 0 \quad \dots(2)$$

and

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\phi z} q = 0 \qquad \dots(3)$$

Since there are two arbitrary constants; (namely a and b) connected by these equations (1) and (2). Therefore, the arbitrary constants a and b can be eliminated. Then, there will appear a relation between x, y, z, p and q in the form

$$f(x, y, z, p, q) = 0 \qquad \ldots(4)$$

which is a partial differential equation of order one.

A solution of a partial differential equation of order one in which the number of arbitrary constants is equal to the number of independent variables is known as the complete integral (C.I.) or complete solution (C.S.) of the partial differential equation. For example, $z = ax + by$, where a and b are arbitrary constants, is the complete integral of the partial differential equation $z = px + qy$.

3.4 *PARTICULAR INTEGRAL:-*

If particular values are given to the arbitrary constants in the complete integral of a partial differential equation of order one, then the solution obtained so, is called the particular integral (P.I) or particular solution (P.S) of the given partial differential equation.

3.5 *SINGULAR INTEGRAL (S.I) OR SINGULAR SOLUTION (S.S):-*

While giving the complete integral (1) of the partial differential equation (4), the supposition was made that a and b are constants and the equation (4) there at was deduced from (1), (2) and (3). But if a and b are assumed to be such functions of the independent variables that these do not alter the forms of p and q, then the partial differential equation given by the elimination of the functions will be the same as in the case when a and b were arbitrary constants, for algebraically elimination takes no account of the value of the quantity eliminated but only of its form.
Now Differentiating $\phi(x, y, z, a, b)$ partially w.r.t. x and y regarding a and b as functions of x and y, we obtain

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \phi}{\partial b}\frac{\partial b}{\partial x} = 0 \qquad \ldots(5)$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial \phi}{\partial b}\frac{\partial b}{\partial y} = 0 \qquad \ldots(6)$$

$\therefore$ The forms of p and q will be the same as in (2) and (3), if we obtain

$$\frac{\partial \phi}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \phi}{\partial b}\frac{\partial b}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial \phi}{\partial b}\frac{\partial b}{\partial y} = 0 \qquad \dots (7)$$

If

$$R = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} \\ \frac{\partial b}{\partial y} & \frac{\partial b}{\partial y} \end{vmatrix}$$

So

$$R\frac{\partial \phi}{\partial a} = 0 \qquad \text{and} \qquad R\frac{\partial \phi}{\partial b} = 0$$

If R is not zero, then

$$\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial b} = 0$$

Thus, the elimination of a and b between the equations $\phi(x, y, z, a, b)$, $\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial b} = 0$ obtains a new solution which is called **Singular Integral or Singular Solution** of the given partial differential equations.

Therefore, singular integral is a relation between the variables involving no arbitrary constant. Sometimes, in extraordinary instances, it happens as a particular integral when special values are given to arbitrary constants appearing in the complete integral, but generally, it is not so and the singular integral (when it exits) is usually distinct from a complete integral.

3.6 *GENERAL INTEGRAL (G.I) OR GENERAL SOLUTION (G.S):-*

In the complete integral $\phi(x, y, z, a, b)$ if the arbitrary constants a and b are functionally associated i.e., if

$$b = \psi(a) \qquad \dots (8)$$

Where ψ is arbitrary function.

Then equation (3) and (4) multiplying by dx and dy, adding another ,we have

$$\frac{\partial \phi}{\partial a} da + \frac{\partial \phi}{\partial b} db = 0 \qquad \dots (9)$$

using (8) in (9), we obtain

$$db = \frac{\partial \psi}{\partial a} da \quad \dots (10)$$

From (10), we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b}\frac{\partial \psi}{\partial a} = 0$$

Hence, this solution is called general integral (G.I) or general solution (G.S) of the given partial differential equation.

Important Notes: A partial differential equation is called fully should when it's all the three types of integrals namely complete integral, singular integral and general integral have been procured otherwise it is not considered fully solved._While solving a non-linear or partial differential equation, we must not only give the complete integral but should also find the singular and general integrals. In absence of details of singular and general integrals, merely the complete integral is considered to be incomplete solution of the given partial differential equation. The students and readers are advised to find the singular and general integrals also for the given partial differential equation, when it is asked to solve the same completely.
Again, when you are asked to find singular and general integrals, then you must find them.

3.7 *GEOMETRICAL INTERPRETATION OF THREE TYPES OF INTEGRALS:-*

i. **Complete Integral:** A complete integral, being a relation between x, y and z, is the equation of a surface. Since it contains two arbitrary parameters, it belongs to a double infinite system of surface or to a single infinite system of family of surfaces.

ii. **General Integral:**
Let a complete integral of $f(x, y, z, a, b) = 0$ be

$$\phi(x, y, z, a, b) = 0 \qquad \dots (1)$$

A general integral is obtained by eliminating a between (1) and the equations

$$b = \psi(a) \qquad \text{... (2)}$$
$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b}\psi' = 0 \qquad \text{... (3)}$$

where ψ is arbitrary function.

The operation of elimination is equivalent to selection of a representative family from the system of families of surfaces and then finding its envelope. The above equations (1), (2), (3) represent a curve drawn on the surface of the family whose parameter is a while the equation obtained by eliminating a between them is the envelope of the family. Consequently, the envelope touches the surface represented by (1) and (2) along the curve be evaluated as by equations (1), (2) and (3). This curve is known as the characteristic of the envelope and the general integral thus represents the envelope of a family of surfaces considered as composed of its characteristics.

iii. Singular Integral:

The singular integral is obtained by eliminating a and b between equations (1) i.e. $\phi(x, y, z, a, b) = 0$ and the equations (1)

$$\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial b} = 0 \qquad \text{... (4)}$$

The operation of elimination is equivalent to find the envelope of all the surfaces included in the complete integral. The above equations (1),(4) give the point of contact of the particular surface regarded by (1) with the general envelope. The singular integral thus represented the general envelope of all the surfaces included in the complete integral.

3.8 *METHOD OF FINDINF SINGULAR INTEGRAL DIRECTLY FROM GIVEN PARTIAL DIFFERENTIAL EQUATION:-*

Let the partial differential equation is

$$f(x, y, z, a, b) = 0 \qquad \text{... (1)}$$

Suppose a complete integral of (1) is written by

$$\phi(x, y, z, a, b) = 0 \qquad \text{... (2)}$$

where a and b are constants.
The singular integral of (1) is obtained by equation (2) and

$$\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial b} = 0 \qquad \dots (3)$$

The values of p and q gained from (2), when substituted in (1) will render it an identity and the replacement of the values of p and q (but not of z) will in general render (1), equivalent to the integral equation. Let this substitution be made so that in (1) p and q are changed by functions of x, y, z, a and b.

Now from (1), we have

$$\frac{\partial f}{\partial p}\frac{\partial p}{\partial a} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial a} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial b} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial b} = 0 \qquad \dots (4)$$

If $\frac{\partial f}{\partial p} \neq 0$ and $\frac{\partial f}{\partial q} \neq 0$, (4) hold if

$$\frac{\partial p}{\partial a}\frac{\partial q}{\partial b} - \frac{\partial p}{\partial b}\frac{\partial q}{\partial a} = 0$$

$$\Rightarrow \qquad \psi(p, q) = 0$$

If both the constants a and b take place in p and q (which does not always happen), the above equation would imply that one of them is a function of the other and the equations using them give general integral which is not now concerned.

Now from (4) are also satisfied, then

$$\frac{\partial f}{\partial p} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial q} = 0 \qquad \dots (5)$$

The elimination of p and q between equations (1) and (5) will delegate a relation between x, y and z independent of any arbitrary constant. If this relation satisfies the differential equation, then it is the singular integral.

3.9 *COMPATIBLE SYSTEM OF FIRST ORDER PDEs:-*

Two partial differential equations

$$f(x, y, z, p, q) = 0 \qquad \ldots (1)$$
$$g(x, y, z, p, q) = 0 \qquad \ldots (2)$$

are called **Compatible**, if they have a common solution.

To find the condition for (1) and (2) to be compatible:

Let

$$J \equiv \frac{\partial(f, g)}{\partial(p, q)} \neq 0 \qquad \ldots (3)$$

Then (1) and (2) can solved to given the explicit expressions for p and q written by

$$p = \phi(x, y, z) \qquad \text{and} \qquad q = \psi(x, y, z) \qquad \ldots (4)$$

The condition that the pair of Eqns. (1) and (2) should be compatible, then, reduces to the condition that the equation

$dz = pdx + qdy$ and $dz = \phi dx + \psi dy - dz = 0$, using (4) ... (5)

should be integrable. The equation (5) is integrable if

$$\phi\left(\frac{\partial\psi}{\partial z} - 0\right) + \psi\left(0 - \frac{\partial\phi}{\partial z}\right) + (-1)\left(\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = 0$$

which is equivalent to

$$\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial y} + \psi\frac{\partial\phi}{\partial z} \qquad \ldots (6)$$

Putting the equations (4) in (1) and differentiating w.r.t. x and z respectively, we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x} = 0 \qquad \ldots (7)$$

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z} = 0 \qquad \ldots (8)$$

From (7) and (8), we have

$$\frac{\partial f}{\partial x} + \phi\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial z}\right) + \frac{\partial f}{\partial q}\left(\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z}\right) = 0$$

Similarly

$$\frac{\partial g}{\partial x} + \phi\frac{\partial g}{\partial z} + \frac{\partial g}{\partial p}\left(\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial z}\right) + \frac{\partial g}{\partial q}\left(\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z}\right) = 0$$

From above two equations are

$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}=\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\} \qquad \dots(9)$$

Again, putting from (4) in (1) and differentiating w.r.t. y and z, we get

$$\frac{\partial\phi}{\partial x}+\psi\frac{\partial\psi}{\partial z}=-\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+\phi\frac{\partial(f,g)}{\partial(z,q)}\right\} \qquad \dots(10)$$

Putting the value of (9) and (10) in (1) and changing ϕ,ψ by p,q respectively, we get

$$\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\}=-\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+\phi\frac{\partial(f,g)}{\partial(z,q)}\right\} \quad \text{or} \quad [f,g]=0$$

where

$$[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}$$

A particular Case: ***To show that first order partial differential equations $\boldsymbol{p=P(x,y)}$ and $\boldsymbol{q=Q(x,y)}$ are compatible if and only if $\boldsymbol{\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}}$.***

Proof: Let the given equations are

$\frac{\partial z}{\partial x}=p=P(x,y)$ and $\frac{\partial z}{\partial y}=q=Q(x,y)$ ... (1)

Since

$$dz=\left(\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial z}{\partial y}\right)dy=pdx+qdy \qquad \dots(2)$$

It follow that the given partial differentials equations (1) are compatible if and only if the single differential equation

$$dz=Pdx+Qdy \qquad \dots(3)$$

is integrable.

$\therefore$ P and Q are functions of two variables x and y, hence $Pdx+Qdy$ is an exact differential if and only if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$. Therefore the equation (3) is integrable if and only if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

Remark1: If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the given partial differentials equations (1) are compatible if and only if the single differential equation. Hence these will possess a common solution.

Remark1: If $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then the given partial differentials equations (1) are compatible if and only if the single differential equation. Hence these will possess no solution.

SOLVED EXAMPLE

EXAMPLE1: Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.

SOLUTION: Let

$$f(x, y, z, p, q) = xp - yq = 0 \qquad \dots (1)$$
$$g(x, y, z, p, q) = z(xp + yq) - 2xy = 0 \qquad \dots (2)$$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & xz \end{vmatrix} = 2xy$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + qy & xz \end{vmatrix} = -x^2p - xyq$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq - 2x & zy \end{vmatrix} = -2xy$$

and

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp + qy & zy \end{vmatrix} = y^2q + xyq$$

$\therefore$

$$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$
$$= 2xy - x^2p^2 - xypq - 2xy + xypq + y^2q^2$$

$= -xp(xp + yq) + yq(xp + yq) = -(xp - yq)(xp + yq) = 0$, using(1)

Hence equations (1) and (2) are compatible.

Solving (1) and (2), we have

$$p = \frac{y}{z}, \qquad q = \frac{x}{z} \quad ...(3)$$

Using (3) in $dz = pdx + qdy$, we obtain

$$dz = \left(\frac{y}{z}\right)dx + \left(\frac{x}{z}\right)dy$$
$$zdz = d(xy)$$

Now integrating,

$$\frac{z^2}{2} = xy + \frac{c}{2} \qquad \text{or} \qquad z^2 = 2xy + c$$

where c is arbitrary constant.

EXAMPLE2: Show that the equations $xp - yq = x$ and $x^2p + q = zx$ are compatible and find their solution.

SOLUTION: Let

$$f(x,y,z,p,q) = xp - yq - x = 0 \qquad ...(1)$$
$$g(x,y,z,p,q) = x^2p + q - zx = 0 \qquad ...(2)$$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp - z & x^2 \end{vmatrix} = (p-1)x^2 - x(2xp - z)$$

Similarly

$$\frac{\partial(f,g)}{\partial(x,p)} = x^2, \quad \frac{\partial(f,g)}{\partial(x,p)} = -q, \qquad \frac{\partial(f,g)}{\partial(x,p)} = -xy$$

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$
$$= (p-1)x^2 - x(2xp - z) - px^2 - q - xyq$$

$= -x^2 + zx - q - xyq = -x^2 + x^2p - qxy$ by (2)

$= x(-x + xp - yq) = 0,$ by (1)

Hence equations (1) and (2) are compatible.

Solving (1) and (2) for p and q, we have

$p = \frac{(1+yz)}{(1+xy)}$ and $p = \frac{x(z-x)}{(1+xy)}$ ... (3)

Using (3) in $dz = pdx + qdy$, we obtain

$$dz = \left[\frac{(1+yz)}{(1+xy)}\right]dx + \left[\frac{x(z-x)}{(1+xy)}\right]dy$$
$$(1+xy)dz = (1+yz)dx + x(z-x)dy$$
$$(1+xy)dz - z(ydx + xdy) = dx - x^2dy$$

$$\frac{(1+xy)dz - zd(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2} = \frac{\left(\frac{dx}{x^2}\right) - dy}{\left(y + \frac{1}{x}\right)^2}$$

$$d\left(\frac{z}{1+xy}\right) = \frac{-d\left(y + \frac{1}{x}\right)}{\left(y + \frac{1}{x}\right)^2}$$

Integrating it,

$$\frac{z}{1+xy} = \frac{1}{\left(y + \frac{1}{x}\right)} + c$$

$$z - x = c(1+xy)$$

where c is arbitrary constant.

3.10 *CHARPIT'S METHOD:-*

Let the given partial differential equation of first order and non-linear in p and q be

$$f(x, y, z, p, q) = 0 \qquad \dots(1)$$

We know that

$$dz = pdx + qdy \qquad \dots(2)$$

We introduce another PDE of the first order of the type

$$F(x, y, z, p, q) = 0 \qquad \dots(3)$$

In order to (3), differentiate partially (1) and (3) w.r.t. x and y and given

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \dots(4)$$

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \dots(5)$$

$$\frac{\partial f}{\partial y} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \dots(6)$$

$$\frac{\partial F}{\partial y} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \dots(7)$$

Eliminating $\frac{\partial p}{\partial x}$ from (4) and (5), we have

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p}-\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p}=0$$

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \quad \dots(8)$$

Similarly Eliminating $\frac{\partial q}{\partial y}$ from (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial q}{\partial y}=0 \quad \dots(9)$$

Since

$$\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

Therefore adding (8) and (9), we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \quad \dots(10)$$

The integral of (10) is given by solving the auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0} \quad \dots(10)$$

Note: In what follows we shall use the following standard notation:

$$\frac{\partial f}{\partial x}=f_x, \frac{\partial f}{\partial y}=f_y, \quad \frac{\partial f}{\partial z}=f_z, \quad \frac{\partial f}{\partial p}=f_p, \quad \frac{\partial f}{\partial q}=f_q$$

Therefore Charpit's auxiliary equations (10) may be written as

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0}$$

SOLVED EXAMPLE

EXAMPLE1: Find the complete integral of $z=px+qy+q^2+p^2$.

SOLUTION: Let

$$f(x,y,z,p,q)\equiv 0z-px-qy-q^2-p^2 \quad \dots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dq}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \quad \dots(2)$$

From (1),

$f_x = -p, f_y = -q, f_z = 1, f_p = -x - 2p, f_q = -y - 2q$ ... (3)

Putting the value of (3) in (1), we get

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \text{ ... (4)}$$

Taking the first fraction of (4), we have

$dp = 0 \quad \Rightarrow p = a$

Taking the second fraction of (4), we have

$dq = 0 \quad \Rightarrow q = a$

Substituting $p = a$ and $q = b$ in (1), the required complete integral is $z = ax + by + a^2 + b^2$, a, b being arbitrary constants.

EXAMPLE2: Find the complete integral of $q = 3p^2$.

SOLUTION: Let the given equation is

$$f(x, y, z, p, q) \equiv 3p^2 - q = 0 \quad \text{... (1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dq}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{... (2)}$$

Or

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{6p^2 + q} = \frac{dx}{-6p} = \frac{dy}{1} \quad \text{using (1)} \qquad \text{... (3)}$$

Taking the first fraction of (3), we have

$dp = 0 \quad \Rightarrow \quad p = a$

Putting the value of p in (1), we obtain $q = 3a^2$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2 dy \quad \Rightarrow \quad z = ax + 3a^2 y + b$ is required complete integral, a and b being arbitrary constants.

EXAMPLE3: Find a complete, singular and general integrals of $(p^2 + q^2)y = qz$.

SOLUTION: Let the given equation is

$$f(x, y, z, p, q) \equiv (p^2 + q^2)y - qz.= 0 \quad \text{... (1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dq}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{... (2)}$$

Or

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dq}{-p^2 y + qz - 2q^2 y} = \frac{dx}{-2py} = \frac{dy}{-2qy + z} \quad \text{by (1)}$$

Taking the first two fractions, we have

$$2pdp + 2qdq = 0 \quad \text{so that} \quad p^2 + q^2 = a \quad \text{... (3)}$$

Using (3), in (1), we obtain

$a^2y = qz$ or $q = \frac{a^2y}{z}$

Substituting the value of q in (3), we have

$$p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \left(\frac{a^4y^2}{z^2}\right)} = \frac{a}{2}\sqrt{z^2 - a^2y^2}$$

Now putting these value in $dz = pdx + qdy$, we obtain

$dz = \frac{a}{2}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dy$ or $\frac{zdz - a^2ydy}{\sqrt{z^2-a^2y^2}} = adx$

Integrating,

$(z^2 - a^2y^2)^{1/2} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$ ... (4)

is required complete integral.

Singular Integral:- Differentiating (4) w.r.t. a and b, we obtain

$$0 = 2ay^2 + 2(ax + b)x \quad \text{... (5)}$$

$$0 = 2(ax + b)x \quad \text{... (6)}$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$

Hence it is singular integral.

General Integral:- changing b by $\phi(a)$ in (4), we have

$$z^2 - a^2y^2 = (ax + \phi(a))^2 \quad \text{... (7)}$$

Differentiating w.r.t. a,

$$-2ay^2 = 2[ax + \phi(a)].[x + \phi'(a)] \quad \text{... (8)}$$

The general integral is obtained by eliminating a from (7) and (8).

EXAMPLE4: Find complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$.

SOLUTION: Here given equation is

$$f(x, y, z, p, q) \equiv 2xz - px^2 - 2qxy + pq = 0. \quad \text{... (1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dq}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2+2xyq-2pq} \quad \text{by (1)}$$

The second fraction obtain $dq = 0$ so that $q = a$

Substituting $q = a$ in (1), we have

$$p = 2x(z - ay)/(x^2 - a)$$

Putting values p and q in $dz = pdx + qdy$, we have given below

$dz = \frac{2x(z-ay)}{x^2-a}dx + ady$ or $\frac{dz-ady}{z-ay} = \frac{2xdx}{x^2-a}$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

Or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$ ... (2)
which is complete integral.
Differentiating (2) w.r.t. a and b, we have
$0 = y - b$ and $0 = x^2 - a$ ... (3)
Solving (3) for a and b, $b = y$ and $x^2 = a$
Putting these values in (2),

$z = x^2 y$ is required the singular solution.

EXAMPLE5: Using Charpit's method, find threecomplete integrals of $pq = px + qy$.

SOLUTION: Here given equation is

$$f(x, y, z, p, q) \equiv pq - px - qy = 0. \qquad ... (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dq}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{-p} = \frac{dq}{-q} = \frac{dx}{-(q-x)} = \frac{dy}{-(p-y)} = \frac{dz}{-p(q-x) - p(p-y)} \quad \text{by (1)} \qquad ... (2)$$

To find first complete integral.
Taking the first two fractions of (2), we have

$\left(\frac{1}{p}\right) dp = \left(\frac{1}{q}\right) dq$ so that $\log p = \log q + \log a$ or $p = aq$ ... (3)

Using (3),

$\Rightarrow \quad aq^2 = q(ax + y) \quad \Rightarrow \quad q = \frac{(ax+y)}{a}$

Hence, from (3), we get

$$dz = pdx + qdy = (ax + y)dx + [(ax + y)/a]dy$$
$$= \left(\frac{1}{a}\right)(ax + y)(adx + y)$$

Substituting $ax + y = t$ so that $adx + dy = dt$,

$\Rightarrow \quad dz = \left(\frac{1}{a}\right) \times tdt$ so that $z = (1/2a) \times t^2 + b$ or $z = (1/2a) \times (ax + y)^2 + b$ as $t = ax + y$.

To find second complete integral. Taking the second and the fourth ratios in (2), we obtain

$\frac{dx}{(q-x)} = \frac{dq}{q}$ or $qdx + xdq = qdq$

Integrating, $qx = \frac{q^2}{2} + \frac{a}{2}$ or $q^2 - 2xq + a = 0$

$\therefore \ q = \left[2x \pm 2(x^2 - a)^{1/2}\right]/2$ so that $q = x + (x^2 - a)^{1/2}$ ... (4)

Using (4), in (1)

$\Rightarrow \ p\left[x + (x^2 - a)^{1/2}\right] - px - y\left[x + (x^2 - a)^{1/2}\right] = 0$

So that

$$p = \left[1 + x/(x^2 - a)^{1/2}\right]y$$

$$dz = pdx + qdy = \left[1 + \frac{x}{(x^2 - a)^{\frac{1}{2}}}\right]ydx + \left[1 + \frac{x}{(x^2 - a)^{\frac{1}{2}}}\right]dy$$

$$dz = (ydx + xdy) + \left[\frac{xy}{(x^2-a)^{\frac{1}{2}}} + (x^2 - a)^{\frac{1}{2}}dy\right] \quad \text{or}$$

$$dz = d(xy) + d\left[y(x^2 - a)^{\frac{1}{2}}\right]$$

Integrating, $z = xy + y(x^2 - a)^{\frac{1}{2}} + b$, a, b being arbitrary constants.

To find third complete integral.

Taking first and fifth ratios of (2) and third complete integral is

$$z = xy + x(x^2 - a)^{\frac{1}{2}} + b.$$

SELF CHECK QUESTIONS

1. Define a complete integral.
2. Define particular integral.
3. What is the difference between singular integral and general integral.

3.11 *SUMMARY:-*

In this unit we have studied the comprehensive view of solving linear first-order PDEs. The complete integral forms the foundation, incorporating arbitrary constants. A particular integral is then determined by satisfying specific conditions. Singular integrals may be encountered in peculiar cases, and Charpit's integral method is a valuable technique for solving such PDEs by exploring characteristic curves and ODEs along them. The general integral combines these components to offer a solution that fulfills the PDE along with any prescribed conditions.

3.12 *GLOSSARY:-*

- **Complete Integral:** The general solution obtained by integrating a first-order linear PDE. It includes arbitrary functions or constants that are determined by additional conditions (e.g., boundary or initial conditions).
- **Particular Integral:** A specific solution obtained from the complete integral by assigning values or functions to the arbitrary constants.

It satisfies both the PDE and any given boundary or initial conditions.

- **Singular Integral:** Integrals that may lead to singular solutions or solutions with singularities. Singular integrals can arise in specific cases or when dealing with certain types of PDEs.
- **General Integral:** The combined solution of a linear PDE, consisting of both the complete integral and any particular integral. It provides a more comprehensive solution that satisfies the PDE and additional conditions.

Charpit's Method: A method for solving first-order linear PDEs by introducing a set of characteristic curves. It involves finding a set of ordinary differential equations (ODEs) along these characteristics, ultimately leading to the determination of the solution.

These terms collectively represent essential concepts in the study and solution of linear first-order PDEs. The complete integral establishes a broad solution space, and the particular integral refines it to meet specific conditions. Singular integrals and Charpit's integral method address peculiar cases and provide alternative approaches to solving these equations. The general integral combines these components for a comprehensive understanding of the solution space.

3.13 *REFERENCES:-*

- Walter A.Strauss(2008), Partial Differential Equations: An Introduction.
- Robert C. McOwen(2011), Partial Differential Equations: Methods and Applications .
- Peter J. Olver (2014), Introduction to Partial Differential Equations.

3.14 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- A.C. King, J. Billingham and S.R. Otto (2003), Differential Equations.
- William F. Trench (2004), Partial Differential Equations: An Introduction with Mathematica and Maple.

3.15 *TERMINAL QUESTIONS:-*

(TQ-1): Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in x, y, z.

(TQ-2): Show that the equation $f(x, y, p, q) = 0$,$g(x, y, p, q) = 0$ are compatible $\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(y,q)} = 0$.

(TQ-3): Show that $\frac{\partial z}{\partial x} = 7x + 8y - 1$ and $\frac{\partial z}{\partial y} = 9x + 11y - 2$ are not Compatible.

(TQ-4): Find the complete integral of $z = pq$.

(TQ-5): Find the complete integral of $(p + y)^2 + (q + x)^2 = 1$.

(TQ-6): Find the complete integral of $z = px + qy + pq$.

(TQ-7): Find the complete integral of $2z + p^2 + qy + 2y^2 = 0$.

(TQ-8): Find the complete integral of $z^2 = pqxy$.

3.16 *ANSWERS:-*

<u>SELF CHECK ANSWERS</u>

1. A complete integral is a solution to a differential equation that includes all possible arbitrary constants. It represents the general solution to the equation, encompassing all possible variations.
2. A particular integral is a specific solution to a differential equation obtained by substituting specific values or functions into the equation. It satisfies the given conditions or boundary conditions of the problem.
3. The difference between a singular integral and a general integral lies in their nature and scope:
   a. **Singular integral:** This refers to an integral that exhibits singular behavior, such as having a singularity at one or more points within its domain. These integrals often require specialized techniques or interpretations to handle the singularities.
   b. **General integral:** This refers to the broader class of integrals that do not necessarily exhibit singular behavior. General integrals can be evaluated using standard integration techniques and do not pose the same challenges as singular integrals in terms of convergence or behavior at specific points.

TERMINAL ANSWERS

(TQ-4): $z = (x + b)(x + a)$

(TQ-5): $z = ax - (1 - a^2)^{1/2}y - xy + b$

(TQ-6): $z = ax + by + ab$

(TQ-7): $2y^2z + y^2(x - a)^2 + y^4 = b$

(TQ-8): $z = ax^a y^{1/a}$

Unit 4: Fundamentals: Classification and Canonical Forms of PDE

CONTENTS:

4.1 *INTRODUCTION:-*

Partial Differential Equations (PDEs) are pivotal in modeling diverse phenomena across scientific disciplines. The classification of PDEs based on order, linearity, and the number of independent variables informs their complexity and solution approaches. Canonical forms in the context of mathematical equations, such as the heat equation, wave equation, and Laplace's equation, serve as standardized representations that simplify the analysis and solution of these equations.

Canonical forms, such as the heat, wave, and Laplace's equations, represent fundamental prototypes with distinct physical interpretations. Understanding these classifications and canonical forms is crucial for choosing appropriate solution methods and unraveling the mathematical and physical intricacies embedded in these equations. Researchers and practitioners utilize analytical, numerical, and computational tools to address challenges posed by PDEs across various scientific domains.

4.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- To express PDEs in standard canonical forms those simplify their analysis.
- To understanding Laplace transformation is to simplify the process of solving linear differential equations.
- Understanding these classifications helps in selecting appropriate solution techniques, studying the well-posedness of problems, and gaining insights into the behavior of physical systems described by these equations. Different solution methods and numerical techniques are often employed depending on the classification of the PDE.
- To Specifying appropriate initial conditions and boundary conditions is essential for solving hyperbolic equations.

4.3 *CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER:-*

Let a general PDEs of second order for a function of two independent variables x and y in the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots (1)$$

Where R, S and T are the continuous functions of x and y only possessing partial derivatives defined in some domains D on the $xy-$plane. Then from (1), we have

i. If $S^2 - 4RT > 0$, the partial differential equation is hyperbolic at a point (x, y) in domain D.

ii. If $S^2 - 4RT = 0$, the partial differential equation is parabolic at a point (x, y) in domain D.

iii. If $S^2 - 4RT < 0$, the partial differential equation is elliptic at a point (x, y) in domain D.

Note: $r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y}, t = \frac{\partial^2 u}{\partial t^2}$.

EXAMPLE: **(i)** Let the one dimensional wave equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ i.e., $r - t = 0$.

Let comparing equation(1), we obtain

$R = 1, S = 0$ and $T = -1$

Hence, $S^2 - 4RT = 0 - \{4 \times 1 \times (-1) \times\} = 4 > 0$

So the given equation is parabolic.

(ii) Let the two dimensional Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ i.e., $r + t = 0$. Let comparing equation(1), we have

$R = 1, S = 0$ and $T = 1$

Hence, $S^2 - 4RT = 0 - \{4 \times 1 \times (1) \times\} = -4 < 0$.So the given equation is elliptic.

(iii) Let the one dimensional diffusion equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$ i.e., $r - q = 0$. Let comparing equation (1), we have

$$R = 1, S = T = 0$$

Hence, $S^2 - 4RT = 0 - \{4 \times 1 \times 0\} = -4 < 0$.So the given equation is parabolic.

4.4 *CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONIN THREE INDEPENDENT VARIABLES:-*

A linear partial differential equation of the second order in 3 independent variables x_1, x_2, x_2 is obtained by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0 \quad \dots (1)$$

Where $a_{ij} (= a_{ji})$, b_i and c are constants or x_1, x_2, x_2 are independent variables and u is dependent variables.

Now $a_{ij} = a_{ji}, A = [a_{ij}]_{3\times3}$ is real symmetric matrix of order 3×3. The eigen values of matrix A are roots of the characteristic equation of A, namely, $|A - \lambda I| = 0$.

Since the equation (1), classified as below:

i. If all eigenvalues of A are non-zero and have the same sign, except precisely one of them, then (1), is called as ***hyperbolic type of equation***.

ii. If $|A| = 0, i.e.,$ any one of the eigen values of A is zero, hen (1), is called as ***perabolic type of equation.***

iii. If all eigenvalues of A are non-zero and of the same sign, except precisely one of them, then (1), is called as ***elliptic type of equation.***

Note: The matrix can be indicated as below:

$$A = \begin{bmatrix} Coeff.of\ u_{xx} & Coeff.of\ u_{xy} & Coeff.of\ u_{xz} \\ Coeff.of\ u_{yx} & Coeff.of\ u_{yy} & Coeff.of\ u_{yz} \\ Coeff.of\ u_{zx} & Coeff.of\ u_{zy} & Coeff.of\ u_{zz} \end{bmatrix}$$

SOLVED EXAMPLE

EXAMPLE1: Classify $u_{xx} + u_{yy} = u_{zz}$ ..(1)

SOLUTION: The matrix of given equation (1), is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since , $|A - \lambda I| = 0., i.e.,$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \quad \text{or} \quad -(1+\lambda)(1-\lambda)^2 = 0$$

Hence, $\lambda = -1,1,1$, the given equation of hyperbolic type.

EXAMPLE2: Classify $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$ ..(1)

SOLUTION: The matrix of given equation (1) is

$u_{xx} + 0.\ u_{xy} + 0.\ u_{xz} + 0.\ u_{yx} + u_{yy} + u_{zz} + 0.\ u_{zx} + u_{yz} + u_{zy} = 0$

$\therefore$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$

Hence, $|A| = 0$, the given equations is of parabolic type.

EXAMPLE3: Classify $u_{xx} + u_{yy} + u_{zz} = 0$ ..(1)

SOLUTION: The matrix of given equation (1) is

$$u_{xx} + 0.\ u_{xy} + 0.\ u_{xz} + 0.\ u_{yx} + 0.\ u_{zx} + 0.\ u_{zy} + 0.\ u_{zx} + u_{yy} + u_{zz} = 0$$

$\therefore$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the eigen values of A are given by $|A - \lambda I| = 0., i.e.,$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$ or $(1-\lambda)^3 = 0$ are giving $\lambda = 1,1,1$

Hence, $\lambda = -1,1,1$, the given equation is of parabolic.

4.5 *CAUCHY'S PROBLEM FOR SECOND ORDER PDES:-*

Consider a second order PDE for the function z, in the independent variables x and y, and let us suppose this equation can be solved clearly for u_{yy} and hence it can be represented y the equation in the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots (1)$$

where R, S and T are the continuous functions of x and y only. If the initial conditions are described along the same curve in the xy −plane, then this problem is called Cauchy problem.

<u>Characteristic Equations And Characteristic Curves:</u>

Corresponding (1), let the λ quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \dots (2)$$

Where $S^2 - 4RT \geq 0$, (2) has real roots. Then

$$\left(\frac{dy}{dx}\right) + \lambda(x, y) = 0 \qquad \dots (3)$$

Are known as characteristic equations.

The equation (3) are known as characteristic curve and simply the characteristics of the second order PDEs(1).

Now, consider the three cases

1. If $S^2 - 4RT > 0$ (if the equation (1) is hyperbolic) , then (2) has two distinct real roots λ_1, λ_2, so that we have two characteristic equations

   $\left(\frac{dy}{dx}\right) + \lambda_1(x, y) = 0$ and $\left(\frac{dy}{dx}\right) + \lambda_2(x, y) = 0$

   Solving it we get two distinct families of characteristics.

2. If $S^2 - 4RT = 0$ (if the equation (1) is parabolic) , then (2) has two equal real roots λ, λ so that we have one characteristic equations

   $$\left(\frac{dy}{dx}\right) + \lambda(x, y) = 0$$

   Solving we get one family of characteristics.

3. If $S^2 - 4RT < 0$ (if the equation (1) is elliptic), then (2) has two complex roots. Hence there are no characteristics. Thus we get two families of complex characteristics when (1) is elliptic.

SOLVED EXAMPLE

EXAMPLE1: Find the characteristics of $y^2 r - x^2 t = 0$.
SOLUTION: Given $y^2 r - x^2 t = 0$ ... (1)
Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$. Here $R = y^2, S = 0$ and $T = -x^2$. Then

$$S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$$

Hence the (1) is hyperbolic everywhere except on the coordinates axes $x = 0$ and $y = 0$.
The λ quadratic is
$R\lambda^2 + S\lambda + T = 0$ or $y^2\lambda - x^2 = 0$ ..(2)
Solving (2), $\lambda = \frac{x}{y}, \ -\frac{x}{y}$. Corresponding characteristic equations are
$\left(\frac{dy}{dx}\right) + (x/y) = 0$ and $\left(\frac{dy}{dx}\right) - (x/y) = 0$
$xdx + ydy = 0$ and $xdx - ydy = 0$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$ is required families of characteristics and these are families of circles and hyperbolas respectively.

EXAMPLE2: Find the characteristics of $x^2 r + 2xys + y^2 t = 0$.

SOLUTION: Given $x^2 r + 2xys + y^2 t = 0$ ... (1)

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$. Here $R = x^2, S = 2xy$ and $T = y^2$. Then

$$S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 4x^2y^2 = 0$$

Hence the (1) is parabolic everywhere.

The λ -quadratic is $R\lambda^2 + S\lambda + T = 0$ or $x^2\lambda^2 + 2xy\lambda + y^2 = 0$ ... (2)

Solving (2), we have

$(x\lambda + y)^2 = 0$ so that $\lambda = -\frac{y}{x}, -\frac{y}{x}$ (equation roots). The characteristic equation is $(dy/dx) - (y/x) = 0$ or $(1/y)dy - (1/x)dx = 0$

Giving $\frac{y}{x} = c_1$ or $y = c_1 x$ is required family of characteristics and it represents a family of straight lines passing through the origin.

4.6 *LAPLACE TRANSFORMATION: REDUCTION TO CANONICAL (OR NORMAL) FORMS:-*

Let the partial differential equation of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

where R, S and T are the continuous functions of x and y only. Laplace transformation on (1) consists of replacing the independent variables x, y to new set of continuously differentiable independent variables u, v where

$u = u(x, y)$ and $v = v(x, y)$ ... (2)

are to be selected so that from (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad \text{... (3)}$$

$$p = \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \quad \text{and} \quad q = \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} \qquad \text{... (4)}$$

From (3) and (4), we get

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

Again (3) and (4), we have

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial u^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and by (3) and (4), we obtain

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

Substituting the above values of p, q, r, s, t in (1), we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \qquad \text{... (5)}$$

where

$$A = R\left(\frac{\partial u}{\partial y}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{... (6)}$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \qquad \text{... (7)}$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad \text{... (8)}$$

And $F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is transformed form of $F(x, y, z, p, q)$.

The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad \dots(9)$$

is everywhere either positive, negative or zero, and we shall present these three cases separately.

Case1: Let $S^2 - 4RT > 0$. When this condition is satisfied, then λ_1, λ_2 of the equation (9) are real and distinct. So the equation (5) will be vanish if

$$\frac{\partial u}{\partial x} = \lambda_1 \left(\frac{\partial u}{\partial y}\right) \qquad \dots(10)$$

$$\frac{\partial v}{\partial x} = \lambda_2 \left(\frac{\partial v}{\partial y}\right) \qquad \dots(11)$$

Since

$$R{\lambda_1}^2 + S\lambda_1 + T = 0 \qquad \dots(12)$$

Now putting the value of (10) in (6), we get

$$A = (R{\lambda_1}^2 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0 \qquad \dots(13)$$

Again

$$R{\lambda_2}^2 + S\lambda_2 + T = 0 \qquad \dots(14)$$

Now putting the value of (11) in (8), we get

$$C = (R{\lambda_2}^2 + S\lambda_2 + T)\left(\frac{\partial v}{\partial y}\right)^2 = 0 \qquad \dots(15)$$

From (10), we obtain

$$(\partial u/\partial x) - \lambda_1(\partial u/\partial y) = 0$$

The Lagrange auxiliary equation is

$$\frac{dx}{x} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

So that $du = 0 \quad \Rightarrow \quad u = c_1, c_2$ are constants.

Taking first and second fraction of above equation, we obtained

$\frac{dy}{dx} + \lambda_1 = 0 \quad \Rightarrow \quad f_1(x,y) = c_1, c_2$ are constants.

The general solution is $u = f_1(x,y), v = f_2(x,y)$

Here f_1 and f_2 are arbitrary function

$$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

$$B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \text{ as } A = C = 0 \quad \dots (16)$$

Let the jacobian J of u and v be non-zero.i.e.,

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0$$

So

$S^2 - 4RT > 0$, (16) prove that $B^2 > 0$. Hence the equation (5), transforms to the form

$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is canonical form of (1) in this case.

<u>CaseII:</u> Let $S^2 - 4RT = 0$, when this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. Take v to be any function of x, y which is independent of u. Now in case I, $A = 0$. Also, $S^2 - 4RT = 0$, (16) shows that$B^2 = 0$. So that$B = 0$.

Moreover $C \neq 0$, Putting $A = 0, B = 0$ and dividing by C, (5) transforms to the form

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{is the canonical form.}$$

<u>CaseIII:</u> Let $S^2 - 4RT < 0$, when this condition is satisfied, the roots λ_1, λ_2 of (9) are complex. Hence this case III is the same as CaseI. Therefore, obtain a real canonical form we make further transformation $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that

$$\alpha = \frac{u+v}{2} \quad \text{and} \quad \beta = \frac{i(u-v)}{2}$$

Now,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{\partial u}\right) + \frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial u}\right) = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right) \quad \dots (17)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{vu}\right) + \frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial v}\right) = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) \quad \ldots (18)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{dv}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$\frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right] = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\frac{\partial^2 z}{\partial \beta \partial \alpha} + i\frac{\partial^2 z}{\partial \beta^2}\right)$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) \quad \text{as} \quad \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \quad \ldots (19)$$

Substituting $u = \alpha + i\beta, v = \alpha - i\beta$ and using(17),(18) and (19), reduce to (16), we get

$$\left(\frac{\partial^2 z}{\partial \alpha^2}\right) + \left(\frac{\partial^2 z}{\partial \alpha^2}\right) = \psi\left(\alpha, \beta, \gamma, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \text{ is canonical form.}$$

4.7 *WORKING RULES:-*

Working rule for reducing a hyperbolic equation to its canonical form:

Step1: Let the given equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \ldots (1)$$

Be hyperbolic so that $S^2 - 4RT > 0$

Step2: Let λ quadratic equation $R\lambda^2 + S\lambda + T = 0$ ... (2)

Step3: Suppose λ_1 and λ_2 be two distinct roots of (2), we have

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \ldots (3)$$

Solving this equation, we obtain, $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$

Step4: Now we select $u = f_1(x, y), v = f_2(x, y)$ ... (4)

where u and v known as characteristic coordinates.

Step5: Using (4), find p, q, r, s and t in terms of u and v.

Step6: putting the values of p, q, r, s and t in (1), we obtain the following canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial u}\right)$$

SOLVED EXAMPLE

EXAMPE1: Reduce $\frac{\partial^2 z}{\partial x^2} = x^2\left(\frac{\partial^2 z}{\partial y^2}\right)$ to canonical form

SOLUTION: Let re-writing, the obtained equation is

$$r - x^2 t = 0 \qquad \dots(1)$$

Now comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$$R = 1, S = 0, T = -x^2$$

$\therefore$ λ- quadratic $R\lambda^2 + S\lambda + T = 0$ written as

$$\lambda^2 - x^2 = 0 \quad \Rightarrow \quad \lambda = \pm x$$

Since $\lambda_1 = x$ and $\lambda_2 = -x$

Hence $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

$\Rightarrow$ $\frac{dy}{dx} + x = 0$ and $\frac{dy}{dx} - x = 0$

Integrating it, $y + \frac{1}{2}x^2 = c_1$ and $y - \frac{1}{2}x^2 = c_2$.

Hence, we change x, y to u, v by taking, we get

$$u = y + \frac{1}{2}x^2 \quad \text{and} \quad v = y - \frac{1}{2}x^2$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\right) = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1.\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values of r and t in (1), we have

$$= x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - \left(\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

Or

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u - v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

Which is required canonical form.

EXAMPE2: Reduce $\frac{\partial^2 z}{\partial x^2} = (1 + y)^2\left(\frac{\partial^2 z}{\partial y^2}\right)$ to canonical form

SOLUTION: Let re-writing, the obtained equation is

$$r - (1 + y^2)t = 0 \qquad \ldots (1)$$

Now comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we obtain

$$R = 1, S = 0, T = -(1 + y)^2$$

$\therefore \quad S^2 - 4RT = (1 + y)^2 > 0$ for $y \neq -1$, showing that (1) is hyperbolic.

λ- quadratic $R\lambda^2 + S\lambda + T = 0$ written as

$$\lambda^2 - (1 + y)^2 = 0 \quad \Rightarrow \quad \lambda = 1 + y, -(1 + y)$$

Hence $\qquad \frac{dy}{dx} + (1 + y) = 0$ and $\frac{dy}{dx} - (1 + y) = 0$

Integrating it,

$\Rightarrow \log(1 + y) + x = C_1$ and $\log(1 + y) - x = C_2$

In order to reduce (1) to its canonical form, we get

$u = \log(1 + y) + x \qquad$ and $\qquad v = \log(1 + y) - x$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial y}\right) = \frac{1}{y+1}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (2)}$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \quad \text{using above equation.}$$

$$r = \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \dots (3)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$
$$+ \frac{1}{1+y}\left[\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)\frac{1}{1+y} + \left(\frac{\partial^2 z}{\partial v^2} + \frac{\partial^2 z}{\partial u \partial v}\right)\frac{1}{1+y}\right]$$

$$t = \frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \qquad \dots (7)$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$4\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}.$$

Working rule for reducing a parabolic equation to its canonical form:

Step1: Let the given equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \dots (1)$$

Be hyperbolic so that $S^2 - 4RT = 0$

Step2: Let λ quadratic equation $R\lambda^2 + S\lambda + T = 0 \qquad \dots (2)$

Step3: Suppose $\lambda_1 = \lambda$ be two equal roots of (2), we have

$$\frac{dy}{dx} + \lambda_1 = 0$$

Solving this equation, we obtain, $f_1(x, y) = c_1$, c_1 being arbitrary constant.

Step4: Now we select $u = f_1(x, y), v = f_2(x, y)$ ... (4)

where u and v known as characteristic coordinates. For this verify Jacobian J of u and v given by (4) is non- zero,

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \neq 0$$

Step5: Using (4), find p, q, r, s and t in terms of u and v.

Step6: putting the values of p, q, r, s and t in (1), we obtain the following canonical form

$$\frac{\partial^2 z}{\partial u^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial u}\right)$$

SOLVED EXAMPLE

EXAMPL1: Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

SOLUTION: Let the given equation is

$$R + 2s + t = 0 \qquad ... (1)$$

be parabolic and comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

$$R = 1, S = 2, T = 1$$

So that

$$S^2 - 4RT = 0$$

Let λ quadratic equation $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1, -1$ (equal roots)

The corresponding characteristic equation is

$$\left(\frac{dy}{dx}\right) - 1 = 0 \qquad dx - dy = 0$$

Integrating, $\quad x - y = c$

where c being constant.

Take, $\quad u = x - y \quad$ and $\quad v = x + y \qquad \ldots(2)$

So

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 1.1 + 1.1 = 2 \neq 0$$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)}$$

From the above equations, we obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)$$

$$= \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

Putting the values of r, s, t in equation (1), we have

$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0$$

Integrating w.r.t. v, we get

$$\frac{\partial z}{\partial v} = \phi(u), \qquad \phi \qquad \text{being arbitrary function.}$$

$$z = \int \phi(u)dv + \psi(u) = v\phi(u) + \psi(u)$$

$z = (x+y)\phi(x-y) + \psi(x-y)$, which is desired solution, ϕ, ψ being arbitrary functions.

EXAMPL2: Reduce the equation $r + 2xs + x^2 t = 0$ to canonical form and hence solve it.

SOLUTION: Let the given equation is

$$r + 2xs + x^2 t = 0 \qquad \dots (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

$$R = 1, S = 2x, T = x^2$$

So that

$$S^2 - 4RT = 0, \text{ showing parabolic.}$$

Let λ quadratic equation $\quad \lambda^2 + 2\lambda x + x^2 = 0 \quad \Rightarrow \quad (\lambda + x^2) = 0$ so that $\lambda = -x, -x$ (equal roots)

The corresponding characteristic equation is

$$\left(\frac{dy}{dx}\right) - x = 0 \qquad dy - xdx = 0$$

Integrating, $\quad y - \frac{x^2}{2} = c_1$

where c_1 being constant.

Take, $\quad u = y - \frac{x^2}{2} \quad$ and $\quad v = x \qquad \dots (2)$

So

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = -1 \neq 0$$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\right) = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial z}{\partial u}, \text{ using (2)}$$

From the above equations, we obtain

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$-\frac{\partial z}{\partial u} - x\left(-x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) - x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$$

$$= \left(x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u}\right)$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$$

Using the values of r, s, t in equation (1), we have

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial u}$$

Which is required canonical form.

<u>Working rule for reducing a elliptic equation to its canonical form:</u>

Step1: Let the given equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots (1)$$

Be hyperbolic so that $S^2 - 4RT < 0$

Step2: Let λ quadratic equation $\quad R\lambda^2 + S\lambda + T = 0 \qquad \ldots (2)$

Step3: Suppose λ_1 and λ_2 be two complex conjugates of (2), we obtain

$$\frac{dy}{dx} + \lambda_1 = 0 \qquad \text{and} \qquad \frac{dy}{dx} + \lambda_2 = 0 \qquad \ldots (3)$$

Solving this equation, we obtain, $f_1(x, y) + if_2(x, y) = c_1$ and $f_1(x, y) - if_2(x, y) = c_2$

Step4: Now we select

$$u = f_1(x, y) + if_2(x, y), v = f_1(x, y) - if_2(x, y) \qquad \ldots (4)$$

where u and v known as characteristic coordinates.Let α and β be two real independent variables such that $u = \alpha + i\beta$ and $v = \alpha - i\beta$

Step5: Using (4), find p, q, r, s and t in terms of u and v.

Step6: putting the values of p, q, r, s and t in (1), we obtain the following canonical form

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

SOLVED EXAMPLE

EXAMPE1: Reduce PDEs $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ or $r + x^2 t = 0$ to canonical form.

SOLUTION: The given equation is

$$r + x^2 t = 0 \qquad \dots (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

$$R = 1, S = 0, T = x^2$$

So that

$$S^2 - 4RT = 0, \text{ showing elliptic.}$$

Let λ quadratic equation $R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda^2 + x^2 = 0$ so that $\lambda = ix, -ix$ (equal roots)

The corresponding characteristic equation is

$$\left(\frac{dy}{dx}\right) + ix = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right) - ix = 0$$

Integrating

$$y + i\left(\frac{x^2}{2}\right) = c_1 \quad \text{and} \quad y - i\left(\frac{x^2}{2}\right) = c_2$$

Select,

$$u = y + i\left(\frac{x^2}{2}\right) = \alpha + i\beta \quad \text{and} \quad u = y - i\left(\frac{x^2}{2}\right) = \alpha - i\beta$$

where $\alpha = y$ and $\beta = \frac{x^2}{2}$ ... (2)

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x}\right) + \frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial x}\right) = x\frac{\partial z}{\partial \beta}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{\partial y}\right) + \frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial y}\right) = \frac{\partial z}{\partial \alpha}, \text{ using (2)}$$

From the above equations, we obtain

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Using the values of $r,,t$ in equation (1), we have
$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$ or $\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}$, as $\beta = \frac{x^2}{2}$

which is required canonical form.

EXAMPE2: Reduce PDEs $y^2\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$ or $y^2 r + x^2 t = 0$ to canonical form.

SOLUTION: The given equation is

$$y^2 r + x^2 t = 0 \qquad \ldots(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

$$R = y^2, S = 0, T = x^2$$

So that

$S^2 - 4RT = 0 = -4x^2y^2 < 0$, for $x \neq 0$, $y \neq 0$, showing elliptic.

Let λ quadratic equation $R\lambda^2 + S\lambda + T = 0 \quad \Rightarrow \quad y^2\lambda^2 + x^2 = 0$ so that $\lambda^2 = -\frac{x^2}{y^2}, \lambda = \frac{ix}{y}, -\frac{ix}{y}$

The corresponding characteristic equation is

$$\left(\frac{dy}{dx}\right) + \frac{ix}{y} = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right) - \frac{ix}{y} = 0$$

Integrating

$$y + ix^2 = c_1 \quad \text{and} \quad y - ix^2 = c_2$$

Select,

$u = y + ix^2 = \alpha + i\beta$ and $u = y - ix^2 = \alpha - i\beta$

where $\alpha = y^2$ and $\beta = x^2$ ... (2)

Since two independent variables

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x}\right) + \frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial x}\right) = 2x\frac{\partial z}{\partial \beta}, \text{ using (2)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\left(\frac{\partial \alpha}{\partial y}\right)+\frac{\partial z}{\partial \beta}\left(\frac{\partial \beta}{\partial y}\right)=2y\frac{\partial z}{\partial \alpha}\text{, using (2)}$$

From the above equations, we obtain

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right)=2\frac{\partial z}{\partial \beta}+2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$=2\frac{\partial z}{\partial \beta}+2x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right]=2\frac{\partial z}{\partial \beta}+4x^2\frac{\partial^2 z}{\partial \beta^2}$$

and

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(2y\frac{\partial}{\partial y}\right)=2\frac{\partial z}{\partial \alpha}+2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial z}{\partial \alpha}$$

$$=2\frac{\partial z}{\partial \alpha}+2y\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial y}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right]=2\frac{\partial z}{\partial \alpha}+4y^2\frac{\partial^2 z}{\partial \alpha^2}$$

Using the values of $r,,t$ in equation (1), we have

$$2y^2\frac{\partial z}{\partial \beta}+4x^2y^2\frac{\partial^2 z}{\partial \beta^2}+2x^2\frac{\partial^2 z}{\partial \alpha^2}+4x^2y^2\frac{\partial^2 z}{\partial \alpha^2}=0$$

or

$$2\alpha\beta\left(\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}\right)+\alpha\frac{\partial z}{\partial \beta}+\beta\frac{\partial z}{\partial \alpha}=0$$

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}+\frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha}+\frac{1}{\beta}\frac{\partial z}{\partial \beta}\right)=0$$

which is required canonical form.

4.8 *SOLUTION OF LINEAR HYPERBOLIC EQUATIONS:-*

It what follow we aim at sketching the existence theorems for two types of initial conditions on the linear hyperbolic equation

$$\frac{\partial^2 z}{\partial x \partial y}=f(x,y,z,p,q) \qquad \dots(1)$$

For both kind of initial conditions, suppose that the function $f(xy,z,p,q)$

Satisfied the following two conditions

i. f is continuous at all points of rectangular region R defined by $\alpha < x < \beta, \gamma < y < \delta$ for all values of x, y, z, p, q concerned.

ii. f satisfied so called Lipschitz condition, mnamely,

$$|f(x, y, z_2, p_2, q_2) - f(x, y, z_1, p_1, q_1)|$$
$$\leq M\{|z_2 - z_1| + |p_2 - p_1| + |q_2 - q_1|\}$$

is all bounded subrectangles r of R.

SELF CHECK QUESTIONS

1. Define a linear hyperbolic partial differential equation.
2. What is the characteristic equation associated with linear hyperbolic equations?
3. What is the Laplace transform of $f(t) = e^{at}? = \frac{1}{s-a}$
4. What is the Laplace transform of $f(t) = sinbt? = \frac{b}{s^2+b^2}$
5. What is the Laplace transform of the unit step function $u(t)? = \frac{1}{s}$
6. What is an elliptic partial differential equation (PDE)?
7. Define the canonical form of an elliptic equation.
8. What are the working rules for reducing an elliptic equation to its canonical form?
9. Define a linear parabolic partial differential equation.
10. What is the purpose of finding characteristic curves for a given PDE?
11. For a first-order linear PDE, what does the characteristic equation.
12. How do you determine the characteristic curves corresponding to a second-order linear PDE in two variables $u(x, y)$.
13. What role do characteristic curves play in determining the type and behavior of solutions to a PDE?
14. In the context of hyperbolic, elliptic, and parabolic PDEs, how do characteristic curves help in classifying these types of equations? represent?

4.9 *SUMMARY:-*

In this unit we have studied In this unit we have studied classified of PDEs based on their order and the number of independent variables, order of a PDE by the highest order of the partial derivatives present in the equation, Canonical forms, Laplace's transformation and solution of hyperbolic function. This unit covered the fundamental aspects of classifying PDEs, understanding their order, transforming them into canonical forms, using Laplace's transformation for solution, and dealing with hyperbolic functions in the context of PDE solutions.

4.10 *GLOSSARY:-*

- **Partial Differential Equation (PDE):** An equation involving partial derivatives that describe how a function depends on multiple variables.
- **Classification of PDEs:** Categorization of PDEs based on their order (first-order, second-order, etc.) and the number of independent variables.
- **Order of a PDE:** The highest order of partial derivatives present in a PDE, determining its classification as first, second, or higher order.
- **Canonical Forms:** Standardized representations of PDEs that simplify analysis and solution by expressing equations in a structured and uniform manner.
- **Linear PDE:** A PDE where the dependent variable and its derivatives appear linearly.
- **Nonlinear PDE:** A PDE where the dependent variable or its derivatives appear nonlinearly.
- **Homogeneous PDE:** A PDE in which every term is a function of the dependent variable and its derivatives and not of the independent variables.
- **Inhomogeneous PDE:** A PDE with terms that depend on both the dependent variable and its derivatives, as well as the independent variables.
- **Canonical Transformation:** A systematic method of transforming a PDE into a standard form or canonical form, making it easier to analyze and solve.
- **Characteristic Curves:** Curves along which PDEs can be transformed into ordinary differential equations (ODEs) in canonical form.
- **Method of Characteristics:** A technique used to find solutions of first-order PDEs by transforming them into systems of ordinary differential equations along characteristic curves.

- **Elliptic PDEs:** PDEs in which the highest-order derivatives are of second order and have mixed partial derivatives.
- **Parabolic PDEs:** PDEs in which the highest-order derivatives are of second order and primarily involve first-order time derivatives.
- **Hyperbolic PDEs:** PDEs in which the highest-order derivatives are of second order and primarily involve second-order spatial derivatives.
- **Laplace Transform:** A mathematical operation that transforms a function of time (often denoted as $f(t)$) into a complex function of a complex variable s (often denoted as $F(s)$).

This glossary covers some fundamental terms related to the classification and canonical forms of PDEs and the Laplace transform, used in engineering and applied mathematics for solving linear differential equations and analyzing dynamic systems

4.11 *REFERENCES:-*

- Sandro Salsa(2008), Partial Differential Equations in Action: From Modelling to Theory.
- Robert C. McOwen(2003), Partial Differential Equations: Methods and Applications.

4.12 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- Michael Shearer and Rachel Levy (2007), Partial Differential Equations: Methods, Applications, and Theories.
- M.D.Raisinghania (2003), Advanced Differential Equations.

4.13 *TERMINAL QUESTIONS:-*

(TQ-1): Classify the following equations:

i. $u_{xx} + u_{yy} = u$

ii. $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + u_{yz}$

(TQ-2): Find the characteristic of $4r + 5s + t + p + q - 2 = 0$.

(TQ-3): Find the characteristic of $4(sin^2x)r + (2cosx)s - t = 0$.

(TQ-4): Reduce the differential equation $t - s + p - q\left(1 + \frac{1}{x}\right) + \frac{z}{x} = 0$ to canonical form.

(TQ-5): Solve $x^2r - y^2t + px - qy = x^2$.

(TQ-6): Reduce $r + 2xs + x^2t = 0$ to canonical form.

(TQ-7): Reduce $r - 6s + 9t + 2p + 3q - z = 0$ to canonical form.

(TQ-8): Reduce $r - 2s + t + p - q = 0$ to canonical form.

(TQ-9): Reduce the following to canonical form and hence solve

i. $x^2r + 2xys + y^2t = 0$

ii. $r - 4s + 4t = 0$

iii. $x^2r + 2xys + y^2t + xyp + y^2q = 0$

iv. $2r - 4s + 2t + 3z = 0$

v. $y^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0$

vi. $z_{xx} + (sech^4 x)z_{yy} = 0$

vii. $y^2\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$

viii. $x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

ix. $3\frac{\partial^2 z}{\partial x^2} + 10\left(\frac{\partial^2 z}{\partial x \partial y}\right) + 3\frac{\partial^2 z}{\partial y^2} = 0$

x. $yr + (x + y)s + xt = 0$

4.14 *ANSWERS:-*

SELF CHECK ANSWERS

3. $\frac{1}{s-a}$

4. $\frac{b}{s^2+b^2}$

5. $\frac{1}{s}$

10. Purpose of finding characteristic curves: The characteristic curves provide a geometric interpretation and insight into the behavior and properties of solutions to the PDE. They help in understanding how solutions propagate and interact within the domain of the PDE.

11. First-order linear PDE and characteristic equation: For a first-order linear PDE, the characteristic equation represents the direction along which characteristics (or curves along which data propagate) are aligned in the solution domain.

12. Determining characteristic curves for a second-order PDE: To determine characteristic curves for a second-order linear PDE in two variables $u(x, y)$ one typically looks for curves along which the coefficients of the highest-order derivatives remain constant. This involves setting the coefficients of the second-order derivatives to zero and solving the resulting system of ordinary differential equations to obtain the characteristic curves.

13. Role of characteristic curves in determining solutions: Characteristic curves guide the propagation of information and determine how solutions evolve over time and space. They help in prescribing initial and boundary conditions and play a crucial role in establishing well-posedness and uniqueness of solutions to PDEs.

14. Classification of PDEs using characteristic curves:

- Hyperbolic PDEs: For hyperbolic equations, characteristic curves correspond to waves along which disturbances propagate. The behavior of solutions is influenced by these characteristic curves.
- Elliptic PDEs: For elliptic equations, characteristic curves typically do not exist in the same sense as in hyperbolic or parabolic cases. The solutions are smooth, and the equation's behavior is characterized by properties such as positivity and coercivity.
- Parabolic PDEs: For parabolic equations, characteristic curves determine the direction of propagation of heat or diffusion. These curves help in understanding the evolution of solutions over time, especially in phenomena governed by diffusion processes.

<u>TERMINAL ANSWERS</u>

(TQ-1):

i. Parabolic
ii. Parabolic

(TQ-2): $y - x = c_1$ and $y - (x/y) = c_2$

(TQ-3): $y + cosecx - cotx = c_1$ and $y + cosecx + cotx = c_2$

(TQ-4): $\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} + \frac{1}{v} \times \frac{\partial z}{\partial u} - \frac{z}{v} = 0$

(TQ-5): $z = \frac{x^2}{4} + \psi(x/y) + \phi(xy)$

(TQ-6): $\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial u}$

(TQ-7): $\frac{\partial^2 z}{\partial v^2} = \frac{z}{9} - \left(\frac{\partial z}{\partial u}\right) + \frac{1}{3} \times \frac{\partial z}{\partial v}$

(TQ-8): $\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial v}, z = \phi(x+y) + e^y \psi(x+y)$

(TQ-9):

i. $z = y\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$
ii. $z = y\phi(y+2x) + \psi(y+2x)$
iii. $z = \phi(y/x) \ + e^{-y}(y/x)$
iv. $z = e^{i\sqrt{3}/2y}\phi(y+x) + e^{-i\sqrt{3}/2y}\psi(y+x)$
v. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\alpha} \times \left(\frac{\partial z}{\partial \alpha}\right)$
vi. $z_{\alpha\alpha} + z_{\beta\beta} = \{2\beta/(1-\beta^2)\}z_\beta$
vii. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0$
viii. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3z}{4} + \frac{3}{4}\frac{\partial z}{\partial \alpha} - \frac{1}{2}\frac{\partial z}{\partial \beta}$
ix. $\frac{\partial^2 z}{\partial u \partial v} = 0, z = f(y-3x) + g(3y-x)$
x. $u^2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0, z = (y-x)^{-1}\psi_1(y^2-x^2) + \psi_1(y-x)$

Unit 5: Monge's Method

CONTENTS:

5.1 *INTRODUCTION:-*

Monge's method, also known as the method of characteristics or the method of lines, is a mathematical technique developed by the French mathematician Gaspard Monge in the late 18th century. This method is primarily used to solve problems related to optimal transportation and allocation of resources. The central problem addressed by Monge's method is often referred to as the transportation problem. In this problem, there are multiple suppliers located at distinct points and multiple consumers also located at distinct points. The task is to find the most cost-effective way to transport a certain amount of resources from the suppliers to the consumers, with each transportation route incurring a specific cost. Monge's method provides a geometric and intuitive approach to solving optimization problems related to transportation. While the method is particularly well-suited for problems with certain structural characteristics, it may not be applicable to all types of transportation and allocation problems. Despite its historical origins, Monge's method remains relevant in fields such as operations research, economics, and logistics.

An equation is said to be of order two, if it involves at least on of the differential coefficients $r = \left(\frac{\partial^2 z}{\partial x^2}\right), t = \left(\frac{\partial^2 z}{\partial y^2}\right), s = r = \left(\frac{\partial^2 z}{\partial x \partial y}\right)$, but now of higher order; the quantities p and q may also enter into the equation. Thus the general form of a second order Partial differential equation is

$$f(x, y, z, q, r, s, t) = 0 \qquad \ldots (1)$$

It is only special cases that (1) can be integrated. Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equation of the form (1). Monge's methods consists in finding one or two first integrals of the form

$$u = \phi(v) \qquad \ldots (2)$$

Where u and v are known as functions of x, y, z, p and q and ϕ is an arbitrary function. In other words, Monge's methods cosnists in obtaining relations of the form (2) such that equation (1) can be derived from (2) by eliminating arbitrary function. A relation of the form (2) is known as an intermediate integral of (1). Every equation of the form (1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (2) as an intermediate integral are of the following forms

$$Rr + Ss + Tt = V \text{ and } Rr + Ss + Tt + U(rt - s^2) = V \qquad \ldots (3)$$

where R, S, T, U and V are functions of x, y, z, p and q.Even equation (3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (3) exist

5.2 *OBJECTIVES:-*

Monge's Method addresses these objectives, offering a comprehensive approach to solving transportation problems and finding applications in various fields where efficient resource management is a critical consideration.

5.3 *MONGE'S METHOD OF INTEGRATING* $\boldsymbol{Rr + Ss + Tt = V}$*:-*

Let the given

$$Rr + Ss + Tt = V \qquad \ldots (1)$$

where R, S, T, U and V are functions of x, y, z, p and q.
We know that

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}$$

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x} & t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y} \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} & s &= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \quad \ldots (2)$$

Now

$$\left.\begin{aligned} dp &= \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy, using\ (2) \\ dq &= \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy, using\ (2) \end{aligned}\right\} \quad \ldots (3)$$

From (3), we get

$r = \frac{dp - sdy}{dx}$ and $r = \frac{dq - sdx}{dy}$

Putting these values of r and s in (1), we obtain

$$r = \frac{dp - sdy}{dx} + Ss + \frac{dq - sdx}{dy} = V$$

$$R(dp - sdy)dy + Ssdxdy + T(dq - sdx)dx = Vdxdy$$

$$(Rdpdy + Tdqdx - Vdxdy) + s(Rdy^2 - Sdxdy + Tdx^2) = 0 \quad \ldots (4)$$

Clearly any relation between x, y, z, p and q which satisfies (5) must also satisfy the following two simultaneous equations

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \ldots (5)$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \quad \ldots (6)$$

The equations (5) and (6) are **called Monge's subsidiary equations** and the relations which satisfy these equations are **called intermediate integrals.**

Now equations (6) being a quadratic, in general, it can be resolved into two equations, say

$$dy - m_1 dx = 0 \quad \ldots (7)$$

$$dy - m_2 dx = 0 \qquad \dots (8)$$

Now the following cases aries:

CaseI: When m_1 and m_2 are distinct in (7) and (8).

In case (5) and (7), if necessary by using well known the result $dz = pdx + qdy$, will gives two integrals $u_1 = a, u_2 = b$, where a, b ar arbitrary constants. These give

$$u_1 = f_1(v_1) \qquad \dots (9)$$

where f_1 is an arbitrary function. It is called an intermediate integral of (1).

Next, taking (5) and (8), we get

$$u_2 = f_2(v_2) \qquad \dots (10)$$

where f_2 is an arbitrary function. Thus we have in two cases two distinct intermediate integrals (9) and (10). Solving these equations, we obtain values of p and q in well known relation

$$dz = pdx + qdy$$

And then integrating (11), is required complete integral of (1).

CaseII: When $m_1 = m_1$ i.e., is perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$$Pp + Qq = R \qquad \dots (11)$$

Solving (11) with the help of Lagrange's method, we obtain the required complete integral of (1).

Remark1: Usually with case I, we get second intermediate integral directly by using symmetry. However sometimes in the absence of any symmetry, we find the complete integral with help of only one intermediate integral. This is done with the help of using Lagrange's method.

Remark2: While getting an intermediate integral, remember to use the relation $dx = pdp + qdy$ as described below

i. $pdx + qdy + 2xdx = 0$ can be re-written as
$dz + 2xdx = 0$ so that $z + x^2 = c$

ii. $xdp + ydq = dx$ can be re-written as
$xdp + ydqpdx + qdy = dx + pdx + qdy$
$d(xp) + d(yq) = dx + dz$ so that $xp + yp = x + z$

Remark3: while integrating, we shall use the following types of calculations. In what follows, f and g are arbitrary functions and k and a are a constant.

i. $\int kf(t)dt = g(t)$
ii. $\int k\frac{1}{t}f(t)dt = g(t)$
iii. $\int k\frac{1}{t^2}f(t^2)d(t^2) = g(t^2)$
iv. $\int kf(x+y)d(x+y) = g(x+y)$
v. $\int kt^2 f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = \int k/\left(\frac{1}{t}\right)^2 f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = g\left(\frac{1}{t}\right)$
vi. $\int \frac{k}{t^2}f(at^2)d(t^2) = \int \frac{k}{(at^2)}f(at^2)d(at^2) = g(at^2)$

Importance Note: For sake of convenience, we have divided all question based on $Rr + Ss + Tt = V$ in four types. We shall now discuss them one by one.

5.4 *WORKING RULE:-*

TypeI based on $\boldsymbol{Rr + Ss + Tt = V}$:

Step1: Write the given equation in the standard form

$$Rr + Ss + Tt = V$$

Step2: Put the values of R, S, T and V in the Monge's subsidiary equations:

$$Rdpdy + Tdqdx - Vdxdy = 0 \qquad \dots(5)$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \qquad \dots(6)$$

Step3: Factories (1) into two distinct factors.

Step4: You should use remark 2, while finding intermediate integral.

Step5: Solve the two intermediate integrals given in step 4 and obtain the values of p and q.

Step6: Putting the values of p and q in $dz = pdx + qdy$ and integrate to arrive at the required general solution. You should use remark 3, while integrating $dz = pdx + qdy$.

SOLVED EXAMPLE

EXAMPLE1: Solve $r = a^2 t$

SOLUTION: Let the given equation

$$r = a^2 t \qquad \dots (1)$$

Comparing it with $Rr + Ss + Tt = V$, we obtain

$R = 1, S = 0, T = -a^2, V = 0$, Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R\,dy^2 - Sdxdy + Tdx^2 = 0$

Since $dpdy - a^2 dqdx = 0, \quad dy^2 - a^2 dx^2 = 0 \quad \dots (2)$

From (20, we get

$$(dy - adx)(dy + adx) = 0$$

Hence two systems of equations are

$$dpdy - a^2 dqdx = 0, \;\; dy - adx = 0 \qquad \dots (3)$$

$$dpdy - a^2 dqdx = 0, \;\; dy + adx = 0 \qquad \dots (4)$$

Now integrating, we have

$$y - ax = c_1$$

Eliminating $\frac{dy}{dx}$ between the equations of (3), we have

$dy - adq = 0$ so that $p - aq = c_2$

Hence,

$$p - aq = \phi_1(y - ax)$$

Similarly

$$p + aq = \phi_2(y + ax)$$

Solving the above equations for p, q, we get

$$p = \frac{1}{2}\{\phi_2(y + ax) + \phi_1(y - ax)\}, \quad q$$
$$= \frac{1}{2a}\{\phi_2(y + ax) - \phi_1(y - ax)\}$$

Putting the values of p, q in $dz = pdx + qdy$, we obtain

$$dz = \frac{1}{2}\{\phi_2(y + ax) + \phi_1(y - ax)\}dx$$
$$+ \frac{1}{2a}\{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$
$$= \frac{1}{2a}\phi_2(y + ax)(dy + adx) - \frac{1}{2a}\phi_2(y - ax)(dy - adx)$$

Integrating again

$z = \psi_2(y + ax) + \psi_1(y - ax), \psi_2, \psi_1$ being arbitrary functions.

EXAMPLE2: Solve $r + (a+b)s + abt = xy$.

SOLUTION: Let the given equation

$$r = a^2 t \qquad \dots (1)$$

Comparing it with $Rr + Ss + Tt = V$, we obtain

$R = t, S = a+b, T = ab, V = xy$, Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R\,dy^2 - Sdxdy + Tdx^2 = 0$

Since

$dpdy + abdqdx - xydqdx = 0,\ dy^2 - (a+b)dxdy + abdx^2 = 0$ ... (2)

From (2), we obtain

$$(dy - bdx)(dy - adx) = 0$$

Hence two systems of equations are

$$dpdy + abdqdx - xydqdx = 0,\ \ dy - bdx = 0 \qquad \dots (3)$$

$$dpdy + abdqdx - xydqdx = 0,\ \ dy - adx = 0 \qquad \dots (4)$$

Now integrating, we have

$$y - bx = c_1 \qquad \dots (5)$$

Eliminating $\frac{dy}{dx}$ between the equations of (3), we have

$dp + abdq - xydx = 0$ or $dp + adq - x(c_1 + bx)dx = 0$

Integrating, $p + aq - \frac{c_1}{2}x^2 - \frac{b}{3}x^3 = c_2$

$$p + aq - \frac{(y - bx)}{2}x^2 - \frac{b}{3}x^3 = c_2$$

$$p + aq - \frac{(y)}{2}x^2 + \frac{b}{6}x^3 = c_2 \qquad \dots (6)$$

Using (5) and (6),the first intermediate integral corresponding to (3), we have

$$p + aq - \frac{(y)}{2}x^2 + \frac{b}{6}x^3 = \phi_1(y - bx)$$

Similarly

$$p + bq - \frac{(y)}{2}x^2 + \frac{b}{6}x^3 = \phi_2(y - ax)$$

Solving above equations, we obtain

$$p = \frac{(y)}{2}x^2 - \frac{1}{6}(a+b)x^3 + (a-b)^{-1}[a\phi_2(y - ax) - b\phi_1(y - bx)]$$

$$q = \frac{1}{6}x^3 + (a-b)^{-1}[\phi_1(y - bx) - \phi_2(y - ax)]$$

Putting these values in $dz = pdx + qdy$

$$dz = \frac{(y)}{2}x^2dx - \frac{1}{6}(a+b)x^3dx$$
$$+ (a-b)^{-1}[a\phi_2(y-ax)dx - b\phi_1(y-bx)dx]$$
$$+\frac{1}{6}x^3dy + (a-b)^{-1}[\phi_1(y-bx)dy - \phi_2(y-ax)dy]$$

$$dz = \frac{1}{6}(3x^2ydx + x^3dy) - \frac{1}{6}(a+b)x^3dx$$
$$-(b-a)^{-1}[\phi_2(y-bx)dx - \phi_1(y-ax)dx]$$
$$-(b-a)^{-1}[\phi_1(y-bx)dy - \phi_2(y-ax)dy]$$
$$dz = \frac{1}{6}(3x^2ydx + x^3dy) - \frac{1}{6}(a+b)x^3dx$$
$$+(b-a)^{-1}\phi_2(y-ax)(dy-adx)$$
$$-(b-a)^{-1}\phi_1(y-bx)(dy-bdx)$$
$$dz = \frac{1}{6}d(x^3y) - \frac{1}{6}(a+b)x^3dx + (b-a)^{-1}\phi_2(y-ax)d(y-ax)$$
$$-(b-a)^{-1}\phi_1(y-bx)d(y-bx)$$

Integrating, $z = \frac{1}{6}(x^3y) - \frac{1}{24}(a+b)x^4 + \psi_2(y-ax) + \psi_1(y-bx)$

Where ψ_1 and ψ_2 are arbitrary functions.

EXAMPLE3: Solve $t - rsec^4y = 2qtany$.

SOLUTION: Let the given equation

$$t - rsec^4y = 2qtany \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we find

$$R = -sec^4y, S = 0, T = 1, V = 2qtany$$

Monge's subsidiary equations are

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{and} \quad R\,dy^2 - Sdxdy + Tdx^2 = 0 \qquad ...(2)$$

Substituting the values of R, S, T and V in above equations, we obtain

$$-sec^4ydpdy + dqdx - 2qtanydxdy = 0 \qquad ...(3)$$
$$-sec^4y\,dy^2 + dx^2 = 0 \qquad ...(4)$$

From (3),

$$(dx - sec^2ydy)(dx + sec^2ydy) = 0$$

So that

$$dx - sec^2ydy = 0 \qquad ...(5)$$
$$dx + sec^2ydy = 0$$

Substituting the value of dx from (5) in (3), we get

$$-sec^4ydpdy + dqsec^2ydy - 2qtanydy.sec^2ydy = 0$$
$$-dp + cos^2ydq - 2qsiny\,cosydy = 0$$
$$dp - (cos^2ydq - 2qsiny\,cosydy) = 0$$
$$dp - d(qcos^2y) = 0$$

Now integrating, $p - q\cos^2 y = c_1$ ... (6)

Integrating (5), we get

$$x - \tan y = c_2 \quad \text{... (7)}$$

From (6) and (7), one integral of (1) s

$$p - q\cos^2 y = f(x - \tan y)$$

Similarly

$$p + q\cos^2 y = g(x + \tan y)$$

Solving above equations for p, q, we find

$$p = \frac{1}{2}(f + g), q = \frac{g - f}{2\cos^2 y} = \frac{1}{2}(g - f)\sec^2 y$$

Now, $dz = pdx + qdy$

$$dz = \frac{1}{2}(f + g)dx + \frac{1}{2}(g - f)\sec^2 ydy$$

$$dz = \frac{1}{2}f(x - \tan y)(dx - \sec^2 ydy) + \frac{1}{2}g(x + \tan y)\ (dx + \sec^2 ydy)$$

$$dz = \frac{1}{2}f(x - \tan y)d(x - \tan y) + \frac{1}{2}g(x + \tan y)\ d(x + \tan y)$$

Integrating it, $z = F(x - \tan y) + G(x + \tan y), F, G$being arbitrary functions.

<u>TypeII based on $\boldsymbol{Rr + Ss + Tt = V}$:</u>

Step1: Write the given equation in the standard form

$$Rr + Ss + Tt = V$$

Step2: Put the values of R, S, T and V in the Monge's subsidiary equations:

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{... (5)}$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \quad \text{... (6)}$$

Step3: Factorise (1) into two distinct factors.

Step4: Taking one factor of step 3 and use (2)to get an intermediate integral. Don't find intermediate integral as we did in type I. If required use remark 1.

Step5: Solve the two intermediate integrals given in step 4 in the form of Lagrange's equation, namely, $Pp + Qq = R$. Using the well known as Lagrange's method we arrive at the desired general solution of the given equation.

SOLVED EXAMPLE

EXAMPLE1: Solve $(r-s)y+(s-t)x+q-p=0$
SOLUTION: The given equation can be given as

$$yr+s(x-y)-tx=p-q \qquad \dots(1)$$

Comparing (1) with $Rr+Ss+Tt=V$, we obtain

$$R=y, S=x-y, T=-x, V=p-q$$

Monge's subsidiary equations are

$Rdpdy+Tdqdx-Vdxdy=0$ and $R\,dy^2-Sdxdy+Tdx^2=0$ ... (2)

$$ydpdy-xdqdx+(q-p)dxdy=0 \quad \dots(3)$$

$$ydy^2-(x-y)dxdy-xdx^2=0 \quad \dots(4)$$

From the above equation

$$(dy+dx)(ydy-xdx)=0$$

$dy+dx=0$ or $dy=-dx$

and $ydx-xdy=0$

Using (3) in (1), we have

$$-ydpdx-xdqdx-(q)dxdx-pdxdy=0$$

$$ydp+xdq+qdx+pdydx=0$$

$$(ydp+pdy)+(xdq+qdx)=0$$

$$d(yp)+d(xq)=0 \quad \Rightarrow \quad yp+qx=c_1 \qquad \dots(5)$$

Integrating it, $x+y=c_2$ ... (6)

From (5) and (6), one intermediate integral is

$$yp+qx=f(x+y)$$

Which is lagrange's form and so

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{f(x+y)}$$

From first and second fractions, we obtain

$$2xdx-2ydy=0$$

Integrating it, $x^2+y^2=a$

Taking second and third fractions, we have

$$\frac{dy}{y}=\frac{dz}{f(x+y)}$$

Or

$$\frac{dx}{(x^2-a)^{1/2}}=\frac{dz}{f[x+(x^2-a)^{1/2}]}$$

$$dv=\frac{x+(x^2-a)^{\frac{1}{2}}}{(x^2-a)^{1/2}}$$

$$\frac{dx}{(x^2-a)^{1/2}} = \frac{dv}{v}$$
$$dz - \frac{1}{v}f(v)dv = 0$$

Integrating again,

$z - F(v) = b$ *or* $z - F\left[x + (x^2-a)^{\frac{1}{2}}\right] = b$

$$z - F(x+y) = b, \quad as \quad y = (x^2-a)^{1/2}$$

Hence, $z - F(x+y) = G(x^2-y^2)$ *or* $z = F(x+y) + G(x^2-y^2)$
where F and G are arbitrary function.

EXAMPLE2: Solve $q(1+q)r - (p+q+2pq)s + p(1+p)t = 0$

SOLUTION: The given equation can be given as

$$q(1+q)r - (p+q+2pq)s + p(1+p)t = 0 \quad \dots(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we obtain

$$R = q(1+q), S = -(p+q+2pq), T = p(1+p)t, V = 0 \quad \dots(2)$$

Monge's subsidiary equations are

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R\,dy^2 - Sdxdy + Tdx^2 = 0$ ... (3)

Using (2) in above equation

$$R\,dy^2 - Sdxdy + Tdx^2 = 0$$
$$(q+q^2)dpdy + (p+p^2)dqdx = 0 \quad \dots(4)$$

and

$$(q+q^2)\,dy^2 + (p+q+2pq)dxdy + (p+p^2)dx^2 = 0 \quad \dots(5)$$
$$= q(1+q)dy^2 + (p+pq)dxdy + (p+pq)dxdy + p(1+p)dx^2$$
$$= q(1+q)dy^2 + p(1+q)dxdy + p(1+q)dxdy + p(1+p)dx^2$$
$$= (1+q)dy(qdy+pdx) + (1+p)dx(qdy+pdx)$$
$$= (1+q)dy(qdy+pdx) + (1+p)dx(qdy+pdx)$$
$$= (qdy+pdx)[(1+q)dy + (1+p)dx] = 0$$

$\Rightarrow \quad qdy + pdx = 0 \quad \Rightarrow \quad qdy = -pdx \quad \dots(6)$

and

$$(1+q)dy + (1+p)dx = 0$$

Putting (6) in (5), we have

$$(1+q)dp(qdy) - (1+p)dq(-pdx) = 0$$

From (6), (qdy) is equal to$(-pdx)$.

Hence, dividing each term of above equation by $(-pdx)$, we obtain

$$(1+q)dp - (1+p)dq = 0$$
$$\frac{dp}{(1+p)} - \frac{dq}{(1+q)} = 0$$

Integrating it, $\log(1+p) - \log(1+q) = logc_1 \quad \dots(7)$

$$\frac{(1+p)}{(1+q)} = c_1 \qquad \dots (8)$$

Using $dz = pdx + qdy$ becomes $dz = 0$ so that $z = c_2$

From (7) and (8), one integral of (10) is

$$\frac{(1+p)}{(1+q)} = f(z) \quad or \quad 1 + p = (1+q)f(z)$$

$$p - f(z)q = f(z)$$

Here the Lagrange's equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{f(z) - 1}$$

Changing 1,1,1 multipliers of each above fractions, we obtain

$$= \frac{dx + dy + dz}{1 - f(z) + f(z) - 1} = \frac{dz}{f(z) - 1}$$

$\Rightarrow \qquad dx + dy + dz = 0$ so that $\quad x + y + z = c_2$

From first and third fraction of Lagrange's equation are

$$dx - (f(z) - 1)^{-1} dz = 0$$

Integrating it, $\quad x + F(z) = c_4$

Hence, $\quad x + F(z) = G(x + y + z), F, G$ being arbitrary functions.

<u>TypeIII based on $Rr + Ss + Tt = V$</u>:

Step1: Write the given equation in the standard form

$$Rr + Ss + Tt = V$$

Step2: Put the values of R, S, T and V in the Monge's subsidiary equations:

$$Rdpdy + Tdqdx - Vdxdy = 0 \qquad \dots (5)$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \qquad \dots (6)$$

Step3: RH.S. of (2) reduces to a perfect square and hence it gives only one factor in place of two as in Type I and Type II.

Step4: Start with only one factor of step 3 and use (2) to obtain an intermediate integral.

Step5: Re-write the intermediate integral of step 4 in the form of $Pp + Qq = R$ and using the well known as Lagrange's method we arrive at the desired the general solution of the given equation.

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Solve $x^2 r - 2xs + t + q = 0$

SOLUTION: Now using Monge's subsidiary equations:

$$x^2 dpdy + dq(-xdy) + q(-xdy)dy = 0 \qquad \dots (1)$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \qquad \dots (2)$$

Now from (1) a, we have

$$(xdy + dx)^2 = 0 \quad \Rightarrow \quad xdy + dx = 0 \quad \dots (3)$$

$$\frac{dx}{x} + dy = 0 \qquad \Rightarrow \qquad y + logx = c_1 \ , \ldots (4)$$

Putting the value of (3) in (1), we obtain

$$x^2 dpdy + dq(-xdy) + q(-xdy)dy = 0$$

$$dp - \left(\frac{dq}{x} - \frac{qdx}{x^2}\right) = 0 \quad or \quad d\left(p - \frac{q}{x}\right) = 0$$

Integrating it,

$$p - \frac{q}{x} = c_2 \qquad \ldots (5)$$

From (4) and (5), we obtain

$$p - \frac{q}{x} - \phi_1(y + logx) \quad or \quad xp - q = x\phi(y + logx)$$

Here the Lagrange's equations are

$$\frac{dx}{x} = \frac{dy}{-1} = \frac{dz}{x\phi_1(y + logx)} \quad \ldots (6)$$

Taking the first two fractions of above equations

$$\frac{1}{x}dx + dy = 0 \quad \Rightarrow \quad y + logx = c_3 \quad \ldots (7)$$

Using(7), first and third fractions of (6) written as

$$\frac{dx}{x} = \frac{dz}{x\phi_1(c_3)} \quad \Rightarrow \quad z - x\phi_1(c_3) = c_4$$

$$z - x\phi_1(y + logx) = c_4$$

Hence

$z - x\phi_1(y + logx) = \phi_2(y + logx)$, ϕ_1, ϕ_2 being arbitrary functions.

EXAMPLE2: Solve $(y - x)(q^2 r - 2pqs + p^2 t) = (p + q)^2(p - q)$

SOLUTION: Now using Monge's subsidiary equations are

$$(y - x)(q^2 dpdy + p^2 dqdx) - (p + q)^2(p - q)dxdy = 0 \qquad \ldots (1)$$

$$q^2 dy^2 + 2pqdxdy + p^2 dx^2 = 0 \qquad \ldots (2)$$

Now from (2) , we have

$$(qdy + pdx)^2 = 0 \quad or \quad qdy + pdx = 0 \qquad \ldots (3)$$

$$dz = pdx + qdy \quad \Rightarrow \quad dz = 0 \ \Rightarrow z = c_1 \qquad \ldots (4)$$

Using (3) in (1), we given that

$$(y - x)(qdp - pdq) - (p + q)^2(p^2 - q^2)(dx - dy) = 0$$

$$q^2 d\left(\frac{p}{q}\right) - \frac{(p^2 - q^2)d(x - y)}{v - x} = 0 \qquad \text{or} \qquad \frac{d(x - y)}{x - y} + \frac{d\left(\frac{p}{q}\right)}{\left(\frac{p}{q}\right)^2 - 1} = 0$$

$$\Rightarrow \qquad \log(x - y) + \frac{1}{2}\log\left(\frac{\left(\frac{p}{q}\right) - 1}{\left(\frac{p}{q}\right) + 1}\right) = \frac{1}{2}logc_2$$

$$\Rightarrow \qquad (x - y)^2 \frac{(p - q)}{p + q} = c_2 \qquad \ldots (5)$$

From (4) and (5), an intermediate integral is

$(x-y)^2 \frac{(p-q)}{p+q} = \phi_1(z)$ or $(x-y)^2(p-q) = (p+q)\phi_1(z)$

$$p\{(x-y)^2 - \phi_1(z)\} - q\{(x-y)^2 + \phi_1(z)\} = 0$$

Here the Lagrange's equations are

$$\frac{dx}{(x-y)^2 - \phi_1(z)} = \frac{dy}{-\{(x-y)^2 - \phi_1(z)\}} = \frac{dz}{0} \quad \dots(6)$$

Now the third fraction of (6) is $dz = 0$ so that $z = a$, where a ia an arbitrary constant.

Taking the first two fractions of above equations

$$\frac{dx+dy}{-2\phi_1(z)} = \frac{dx-dy}{2(x-y)^2}$$

$$d(x+y) = \phi_1(a)\frac{d(x-y)}{(x-y)^2}$$

Integrating, $x + y - \phi_1(a)(x-y)^{-1} = b$

$$x + y - \phi_1(z)(x-y)^{-1} = b \quad \dots(7)$$

From (7) and (6), written as

$$x + y - \phi_1(z)(x-y)^{-1} = \phi_2(z)$$

where ϕ_1, ϕ_2 being arbitrary functions.

TypeIV based on $Rr + Ss + Tt = V$:

Let R.H.S. of $Rdy^2 - Sdxdy + Tdx^2 = 0$ neither gives two factors nor a perfect square (TypeI, TypeII, TypeIII above). In such cases factors $dx, dy, p, 1+p$ etc. are cancelled as the case may be and an integral of given equation is given as usual. This integral is then integrated by methods described in previous chapter.

SOLVED EXAMPLE

EXAMPLE1: Solve $(q+1)s = (p+1)t$

SOLUTION: Given $(q+1)s - (p+1)t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, we have

$$R = 0, \quad S = q+1, \quad T = -(p+1), \quad V = 0 \quad \dots(2)$$

Monge's subsidiary equations are

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \dots(3)$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \quad \dots(4)$$

Putting the values of (2), in (3) and (4), we get

$$-(1+p)dpdx = 0 \quad \dots(5)$$

$$-(q+1)dxdy - (p+1)dx^2 = 0 \quad \dots(6)$$

Dividing (5) by $-(1+p)dx$ and (6) by $-dx$ we get

$$dq = 0$$

$$(q+1)dy + (p+1)dx = 0$$

$$dy + qdy + pdx + dx = 0$$

$dx + dy + dz = 0$ where $dz = pdx + qdy$

Integrating it, $x + y + z = c_1, \quad q = c_2$ ... (7)

From (7),an integral of (10) is

$q = f(x + y + z)$ or $\frac{\partial z}{\partial y} = f(x + y + z)$

Integrating above equation w.r.t. y (treating x as constant), we get

$z = F(x + y + z) + G(x), F, G$ being arbitrary functions.

EXAMPLE2: Solve $pq = x(ps - qr)$.

SOLUTION: Given $xqr - xps + 0.t = -pq$ ... (1)

Monge's subsidiary equations are

$$xqdpdy + pqdydx = 0 \quad \text{... (3)}$$

$$xqdy^2 + xpdxdy = 0 \quad \text{... (4)}$$

Dividing (2) by qdy and (3) by xdy, we have

$$xdp + pdx = 0 \quad \text{... (5)}$$

$$qdy + pdx = 0 \quad \text{... (6)}$$

Now using $dz = pdx + qdy$, so $dz = 0 \quad \Rightarrow z = c_1$

Again integrating, $xp = c_2$ ... (7)

From (6) and (7), we obtain one integral of (1) is

$$xp = f(z) \quad \Rightarrow \quad x\frac{\partial z}{\partial x} = f(z) \quad \Rightarrow \quad \frac{1}{f(z)}\frac{\partial z}{\partial x} = \frac{1}{x}$$

Integrating it partially w.r.t. x we have

$$F(x) = logx + G(y).$$

5.5 *MONGE'S METHOD OF INTEGRATING THE EQUATION* $\boldsymbol{r + Ss + Tt + U(rt - s^2) = V}$*:-*

Given $R\,r + Ss + Tt + U(rt - s^2) = V$ ... (1)

Let, we get

$$dp = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}dy = rdx + sdy$$

and

$$dq = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}dy = sdx + tdy$$

So

$$r = \frac{dp - sdy}{dx} \quad and \quad t = \frac{(dq - sdx)}{dy}$$

Substituting these values in (1), and simplifying, we have

$$(Rdpdy + Tdqdx - Vdxdy - Udpdq) - (Rdy^2 - Sdxdy + Tdx^2 + Udpdx + Udqdy) = 0$$

Now Monge's subsidiary equations are

$$L \equiv Rdpdy + Tdqdx - Vdxdy - Udpdq = 0 \qquad \dots (2)$$

$$M \equiv Rdy^2 - Sdxdy + Tdx^2 + Udpdx + Udqdy = 0 \qquad \dots (3)$$

We cannot factorise Mas we did before, on around of the presence of the additional term $Udpdx + Udqdy$**.** Hence let us factorise $M + \lambda L$, where λ is some multiplier to be determined later. Now we obtain

$$M + \lambda L \equiv Rdy^2 - (S + \lambda V)dxdy + Tdx^2 + Udpdx + Udqdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq = 0 \qquad \dots (4)$$

Comparing coefficients in (4) and (5), we can be written as

$$M + \lambda L \equiv (Rdy + mTdx + kUdp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0 \quad \dots (5)$$

Since

$$\frac{R}{m} + mT = (S + \lambda V) \qquad \dots (6)$$

$$k = m \text{ and } \frac{R\lambda}{k} = U \qquad \dots (7)$$

Now, the two relations of (7) give $m = R\lambda U$

Substituting this value of m in (6) and simplifying, we have

$$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \dots (8)$$

Which is quadratic in λ. Let λ_1 and λ_2 be it roots.

When $= \lambda_1$, we obtain $\quad m = R\lambda_1/U$.

From (5), we get

$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

$$(Udy + \lambda_1 Tdx + U\lambda_1 dp)(R\lambda_1 dy + Udx + U\lambda_1 dq) = 0 \quad \dots (9)$$

Similarly for $\lambda = \lambda_2$, putting from (5), we have

$$(Udy + \lambda_2 Tdx + U\lambda_2 dp)(R\lambda_2 dy + Udx + U\lambda_2 dq) = 0 \quad \dots(10)$$

Now one factor of (9) is combined with one factor of (1) to give an intermediate integral. Exactly similarly the other pair will give rise to another intermediate integral. In this connection remember that we must combine first factor of (9) with the second factor of (10) and similarly the second factor of (9) with the first factor of (10). Thus for the desired solution the propose method is so combine the factors in the following manner:

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + U\lambda_1 dp = 0 \\ Udx + \lambda_2 Rdy + U\lambda_2 dq = 0 \end{aligned}\right\} \quad \dots(11)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + U\lambda_2 dp = 0 \\ Udx + \lambda_1 Rdy + U\lambda_1 dq = 0 \end{aligned}\right\} \quad \dots(12)$$

Since the equation (11) give two integrals $u_1 = c$ and $v_1 = d_1$ so that one intermediate integral is given by

$$u_1 = f_1(v_1) \quad \dots(13)$$

Similarly

$$u_2 = f_2(v_2) \quad \dots(14)$$

Since now solve (13) and (14) for p and q and put in $dz = pdx + qdy$, which after integration gives the desired general solution.

Remark1: There are in all four ways of combining factors of (9) and (10). By combining the first factors in these equations, we would get $udy = 0$ on subtraction (after dividing equations by λ_1 and λ_2 respectively) and this would not produce any solution. Similarly, combining the second factors in these equations would give $udx = 0$ and hence would produce no solution. Hence for getting integrals of the given equation we must proceed as explained in (11) and (12).

Remark2: In what follows we shall use the following two results of the λ quadratic $a\lambda^2 + b\lambda + c = 0$.

i. $a = b = 0, i.e.,$ the coefficient of λ^2 and λ both equal to zero imply that roots of the equation are equal to ∞.

ii. $a = 0$ but $b \neq 0 i.e.,$ the coefficient of λ^2 is zero but that of λ is non-zero imply that one root of the equation is ∞ and the other is $-\frac{c}{b}$.

Remark3: when the two values of λ are equal, we shall have only one intermediate integral $u_1 = f_1(v_1)$ and proceed as explained in solved examples of Type I based on $R\,r + Ss + Tt + U(rt - s^2) = V$ written below. An integral of a more general form can be given by taking the arbitrary function occurring in the intermediate integral to be linear. Let $u_1 = mv_1 + n$, where m and n are some constants. Then integrating it by Lagrange's method we find the solution of the obtained equation.

<u>Solved Example</u>

EXAMPLE1: Solve $5r + 6s + 3t + 2(rt - s^2) + 3 = 0$.
SOLUTION: Given $5r + 6s + 3t + 2(rt - s^2) = -3$ ... (1)
Comparing the given equation with $R\,r + Ss + Tt + U(rt - s^2) = V$, we obtain
$R = 5, S = 6, T = 3, U = 2$ and $V = -3$. Hence

$$\lambda^2(UV + RT) + \lambda SU + U^2 = 0$$

Becomes $9\lambda^2 + 12\lambda + 4 = 0$ or $(3\lambda + 2)^2 = 0$ so that $\lambda_1 = \lambda_2 = -\frac{2}{3}$.
Hence

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + U\lambda_1 dp = 0 \\ Udx + \lambda_2 Rdy + U\lambda_2 dq = 0 \end{aligned}\right\} \quad \text{... (1)}$$

$$2dy + \left(-\frac{2}{3}\right).3dx + \left(-\frac{2}{3}\right).2dp = 0$$

and $$2dy + \left(-\frac{2}{3}\right).3dx + \left(-\frac{2}{3}\right).2dp = 0$$

$3dy - 3dx - 2dp = 0$ and $3dx - 5dy - 2dq = 0$
Integrating, $3y - 3x - 2p = c_1$ and $3y - 5x - 2q = c_2$ ... (2)
Here the only one intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q) \quad \text{... (3)}$$

where f is an arbitrary function.
From (2), we obtain
$p = \frac{1}{2}(3y - 3x - c_1)$ and $q = \frac{1}{2}(3x - 5y - c_2)$
Substituting the value of p and q in $dz = pdx + qdy$, we get

$$dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1 dx - c_2 dy$$

$$2dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1 dx - c_2 dy$$

Integrating, $2z = 3xy - \frac{3}{2}x^2 - \frac{5}{2}y^2 - c_1 x - c_2 y + c_3$

which is required the complete integral, c_1, c_2, c_3 being arbitrary constants.
EXAMPLE2: Solve $3r + 4s + t + (rt - s^2) = 1$.
SOLUTION: Given $5r + 6s + 3t + 2(rt - s^2) = -3 \quad \dots (1)$
Comparing the given equation with $R\,r + Ss + Tt + U(rt - s^2) = V$, we obtain
$R = 3, S = 4, T = 1, U = 1, V = 1$, Hence

$$\lambda^2(UV + RT) + \lambda SU + U^2 = 0$$

Becomes $4\lambda^2 + 4\lambda + 1 = 0$ or $(2\lambda + 1)^2 = 0$ so that $\lambda_1 = \lambda_2 = -\frac{1}{2}$.
Now there is only one intermediate integral obtained by equations

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + U\lambda_1 dp = 0 \\ Udx + \lambda_2 Rdy + U\lambda_2 dq = 0 \end{aligned}\right\} \quad \dots (1)$$

$$dy + \left(-\frac{1}{2}\right)dx + \left(-\frac{1}{2}\right)dp = 0 \quad \text{and} \quad dx + \left(-\frac{1}{2}\right).3dy + \left(-\frac{1}{2}\right)dq = 0$$

$$-2dy + dx + dp = 0 \text{ and } 2dy - 2dx + dq = 0 \quad \dots (2)$$

Hence the only one intermediate integral is obtained by

$$-2y + x + p = f(3y - 2x + q)$$

where f is an arbitrary function.
Now from (2), we have

$$p = 2y - x + c_1, q = -3y + 2x + c_2$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

$$dz = 2(ydx + xdy) - xdx + 3ydy + c_1dx + c_2dy$$

Integrating $z = 2xy - \frac{1}{2}x^2 - \frac{3}{2}y^2 + c_1x + c_2y + c_3$ is required complete integral, c_1, c_2, c_3 being arbitrary constants.

SELF CHECK QUESTIONS

1. What do characteristic curves represent in the context of Monge's method?
2. Why are initial conditions or boundary conditions necessary when using Monge's method? Explain.
3. Compare Monge's method with other methods used to solve first-order linear PDEs. Highlight advantages and limitations.
4. What are the characteristics in Monge's method?
5. Can Monge's method be applied to nonlinear partial differential equations?

5.6 *SUMMARY:-*

Monge's method, especially when expressed through PDEs, poses challenges in terms of analytical solutions. In many cases, numerical methods are employed to approximate solutions. Additionally, the original Monge formulation can be restrictive, leading to the introduction of Kantorovich's relaxed formulation, which is often more amenable to computational methods. Monge's method and its associated PDEs have applications in diverse fields, such as optimal resource allocation, economics, and image processing, where understanding the optimal transport of mass or information is crucial.

5.7 *GLOSSARY:-*

- **Monge's Method:** A technique for solving first-order partial differential equations (PDEs) using characteristic curves. Also known as the method of characteristics.
- **Partial Differential Equation (PDE):** An equation that involves partial derivatives of a function with respect to two or more independent variables. Describes the relationship between the function and its partial derivatives.
- **Method of Characteristics:** A mathematical technique that involves finding characteristic curves along which the solution to a PDE remains constant. Particularly effective for first-order PDEs.
- **Characteristics:** Curves along which the solution to a PDE remains constant. Determined by solving a system of ordinary differential equations (ODEs) derived from the PDE.
- **Parameterization:** The process of introducing auxiliary variables to represent the characteristics. This helps in transforming the PDE into a system of ordinary differential equations (ODEs) along the characteristics.
- **Ordinary Differential Equation (ODE):** An equation involving derivatives of a function with respect to a single independent variable. Characteristic equations along the characteristics are often ordinary differential equations.
- **Reduction to ODEs:** The step in Monge's method where the original PDE is transformed into a set of ordinary differential equations along the characteristic curves.
- **Compatibility Conditions:** Conditions that must be satisfied by the solutions along different characteristics to ensure the overall consistency of the solution to the PDE.
- **Integration:** The final step in Monge's method involves integrating the ordinary differential equations along the characteristic curves to determine the solution to the original partial differential equation.

5.8 *REFERENCES:-*

- Andrea Bressan(2002)Hyperbolic Partial Differential Equations.
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5.9 *SUGGESTED READING:-*

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- David Borthwick (2016), Introduction to Partial Differential Equations.
- Michael Shearer, Rachel Levy (2017),Partial Differential Equations: An Introduction to Theory and Applications.
- http://ddugu.ac.in/ePathshala_Attachments/chapter-9.pdf
- http://www.neo-classical-physics.info/uploads/3/4/3/6/34363841/zervos_-_the_monge_problem.pdf
- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.

5.10 *TERMINAL QUESTIONS:-*

(TQ-1): Solve $r = a^2t$

(TQ-2): Solve $r = t$

(TQ-3): Solve $r - tcos^2x + ptanx = 0$

(TQ-4): Solve $(r - s)y + (s - t)x + q - p = 0$

(TQ-5): Solve $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0$

(TQ-6): Solve the following by Monge's method.

i. $(x - y)(xr - xs - ys + yt) = (x - y)(p - q)$
ii. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$
iii. $x^2r - y^2t - 2xp + 2z = 0$
iv. $(r - t)xy - s(x^2 - y^2) = qx - py$
v. $q^2r + p^2t - 2pqs = pt - qs$
vi. $x^2r - y^2t = xp - yq$

(TQ-7): Obtain the integral $q^2r + p^2t - 2pqs = 0$ in the form $y + xf(z) = F(z)$

(TQ-8): Explain the fundamental principles of Monge's method and its role in solving partial differential equations. Provide a step-by- step walkthrough of the application of Monge's method to a generic partial differential equation.

(TQ-9): Apply Monge's method to integrate the partial differential equation $R\,r + Ss + Tt + U(rt - s^2) = V$. Provide a step-by-step explanation of how Monge's method is employed to find the solution, and discuss any considerations or special cases in the process.

(TQ-10): Apply Monge's method to integrate the partial differential equation $R\,r + Ss + Tt = V$. Provide a detailed walkthrough of the steps involved, including the derivation of characteristic equations, solving ordinary differential equations, and reconstructing the solution. Discuss any considerations or special cases that arise during the process.

5.11 *ANSWERS:-*

SELF CHECK ANSWERS

1. Characteristic curves represent paths along which the solution to a partial differential equation remains constant.
2. Initial or boundary conditions are necessary to determine the specific solution from the general solution obtained through the method of characteristics.
3. Monge's method is particularly effective for problems with characteristics that are easy to compute, but it may become complex for certain scenarios. Other methods, like the method of characteristics or separation of variables, may be more suitable in different situations.
4. Characteristics are curves along which the partial differential equation reduces to an ordinary differential equation.
5. Yes

TERMINAL ANSWERS

(TQ-1):$z = \psi_2(y + ax) + \psi_1(y - ax)$

(TQ-2):$z = \psi_2(y + x) + \psi_1(y - x)$

(TQ-3):$z = F(y - sinx) + G(y + sinx)$

(TQ-4):$z = F(x + y) + G(x^2 - y^2)$

(TQ-5):$x + F(z) = G(x + y + z)$

(TQ-6):

- **i.** $F(xy) + z = G(x + y)$
- **ii.** $z - xy + F(x^2 + y^2) = G\left(\frac{y}{x}\right)$
- **iii.** $zy + (xy)^{\frac{3}{2}}\psi_1\left(\frac{y}{x}\right) = \psi_2(yx)$

iv. $z = F(x^2 + y^2) + G\left(\frac{y}{x}\right)$

v. $y = F(z) + G(x - z)$

vi. $z = x^2 H\left(\frac{y}{x}\right) + G(xy)$

(TQ-7): $y + f(xz) = F(z)$

BLOCKII
PARTIAL DIFFERENTIAL EQUATIONS-II

Unit 6: Laplace and Poisson Equations

CONTENTS:

6.1 *INTRODUCTION:-*

In this unit we will study about the Laplace's and Poisson's equations are fundamental in mathematical physics, finding applications in diverse fields such as electromagnetism, fluid dynamics, and structural mechanics. Solving these equations provides insights into the steady-state behavior of physical quantities and is crucial in understanding and designing various engineering and scientific systems.

6.2 *OBJECTIVES:-*

The objectives of studying Laplace's equation and Poisson's equation are broad, encompassing both theoretical understanding and practical applications in fields ranging from physics and engineering to mathematics. The knowledge gained from these studies is foundational for solving real-world problems and advancing scientific and technological innovations.

6.3 *LAPLACE EQUATION AND ITS SOLUTION:-*

The Laplace equation is a partial differential equation that describes the distribution of a scalar field in space. It is named after the French mathematician Pierre-Simon Laplace, who made significant contributions to the field of mathematical physics. The Laplace equation is often denoted as:

$$\nabla^2 \phi = 0$$

Here, ∇^2 delta represents the Laplacian operator, which is the diverence of the gradient of a scalar field ϕ. In **three-dimensional Cartesian coordinates**, the Laplace equation can be expressed as:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The Laplace equation is a special case of the more general Poisson equation when the source term is zero.

Obtaining Poisson's equation is exceedingly simple, for from the point form ofGauss's law,

$$\nabla . D = \nabla . \in E = \rho_v \qquad \ldots (1)$$

And $$E = -\nabla V \qquad \ldots (2)$$

Putting the equation (2) into equation (1) obtains

$$\nabla . (-\in \nabla V) = \rho_v \qquad \ldots (3)$$

for an inhomogeneous medium. For a homogeneous medium, equation (3) gives

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\in} \qquad \ldots (4)$$

This is called **Poisson's equation.** Then the equation (4) is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \ldots (5)$$

which is called **Laplace's equation and** ∇ is called Laplace's operator. Note that in taking ($\in$) out of the left-hand side of equation (3) to given equation (4), we have acquired that ($\in$) is constant throughout the region in which V is defined; for an inhomogeneous region, (∈) is not constant and equation (4) does not follow equation (3). Equation (3) is Poisson's equation for an inhomogeneous medium; it gives Laplace's Equation for an homogeneous medium when

$$\rho_v = 0$$

Thus Laplace's equation in Cartesian, Cylindrical, spherical coordinates r is given below

$$\nabla^2 . V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(Cartesian Coordinate)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(Cylindrical Coordinate)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

(Spherical Coordinate)

depending on whether the potential is $V(x, y, z), V(\rho, \phi, z)$ or $V(r, \theta, \phi)$.

Apart from the contexts previously mentioned, Laplace's and Poisson's equations have applications across various disciplines and phenomena. Some additional instances Laplace's and Poisson's equations include the following.

1. **Electrostatics:** From Maxwell's equations, one has $curlE = 0$ and $divE = 4\pi\rho$, where $\rho$is the charge density. The first equation implies $E = -gard\phi$ for a scalar function φ (known as electric potential). Therefore,

$$\Delta\phi = div(gard\phi) = -divE = -4\pi\rho$$

∴ which is poisson's equation $(f = -4\pi\rho)$.

2. **Steady fluid flow**: Suppose that the flow is irrotational (no eddies) so that $curlV = 0$, where $V = v(x, y, z)$ is the velocity at the position (x, y, z), Let us consider independent of time. Assume that the fluid is incompressible (e.g., water) and that there are no sources or sinks. Then $divV = 0$. Hence $V = -gard\phi$ for some ϕ (known as velocity potential) and $\Delta\phi = -divV = 0$,which is Laplace's equation.

3. **Analytic functions of a complex variable:** Let we write $z = x + iy$ and $f(z) = u(z) + iv(z) = u(x + iy) + iv(x + iy)$, where u and v are real-valued functions. An analytic function is one that is

explainable as a power series in z. This means that the powers are not$x^n y^n$ but $z^n = (x + iy\,)n$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$ie.,$

$$u(x + iy\,) + iv(x + iy) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

Now this series show that,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (These are the Cauchy–Riemann equations)

Hence we find that

$$u_{xx} = v_{yx} = v_{xy} = u_{yy}$$

So

$$\Delta u = 0$$

Similarly

$$\Delta v = 0$$

where Δ is two dimensional Laplace equation. So the real and imaginary parts of an analytic function are harmonic.

4. **Brownian motion**: Imagine Brownian motion in a container D. This means that particles inside D move completely randomly until they hit the boundary, when they stop. Divide the boundary arbitrarily into two pieces, C_1 and C_2 . Let $u(x, y, z)$ be the probability that a particle that begins at the point (x, y, z) stops at some point of c_1. Then it can be concluded that

$$\Delta u = 0 \text{ in } D$$

$u = 1$ on C_1, $u = 0$ on C_2.

Thus u is the solution of a Dirichlet problem.

6.4 *HARMONIC FUNCTION:-*

A function $\phi(x, y, z)$ is called **Harmonic** at a point (x, y, z) if its second partial derivatives exist, are continuous, and satisfy Laplace's equation $\nabla^2 V = 0$ in the neighborhood of that point. If ψ is harmonic at every point within a domain (or open continuum), it is termed harmonic in that domain. Furthermore, if ϕ is harmonic at all interior points of a closed region, it is considered harmonic within that closed region. The conditions for harmonicity encompass the smoothness of the function and the fulfillment of Laplace's equation across the specified spatial domain. A function $\phi(x,y,z)$ is called at infinity, $\phi r, r^2\frac{\partial\phi}{\partial x}, r^2\frac{\partial\phi}{\partial y}, r^2\frac{\partial\phi}{\partial z}$ bounded for r where $r^2 = x^2+y^2+z^2$.

If the function ψ is harmonic in an unbounded region, then it must be regular at infinity.

<u>Properties of Harmonic function:</u>

i. If a harmonic function vanishes everywhere on the boundary of a domain, then it is identically zero everywhere.
ii. If a function ψ is harmonic in V and $\frac{\partial\psi}{\partial n} = 0$ on S, then ψ is constant in V .
iii. If the Dirichlet problem for a bounded region has a solution, then it is unique.
iv. If the Neimann problem for a bounded region has a solution, then it is either unique or differs from one another by a constant.

6.5 *UNIQUENESS THEOREM:-*

If a solution of Laplace's equation satisfies a obtained set of boundary conditions, there is only one solution. We say that the solution is unique. Thus any solution of Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used. This is called **Uniqueness Theorem**. We acquire that there are two solutions and of Laplace's equation both of which satisfy the prescribed boundary conditions. Thus

$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ ... (1)

$$V_1 = V_2 \quad \dots (2)$$

we assume that

$$V_d = V_2 - V_1 \quad \dots (3)$$

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \quad \dots (4)$$

$V_d = 0$ on the boundary

According to equation (1) and (2), from the divergence theorem

$$\int_V \nabla . A \, dV = \oint_S A . ds \qquad \dots (5)$$

Let $A = V_d . \nabla V_d$ and use the vector identity

$$\nabla . A = \nabla . [V_d \nabla V_d] = V_d [\nabla^2 V_d] + \nabla V_d . \nabla V_d$$

But $\nabla^2 V_d = 0$, according to (4) and (5), we have

$$\nabla . V = \nabla V_d . \nabla V_d \qquad \dots (6)$$

Putting (6) into (5) obtain

$$\int_V \nabla V_d . \nabla V_d \;\, dV = \oint_S V_d . \nabla V_d \, . ds \qquad \dots (7)$$

From the above equation, it is evident that the right hand side of (7) vanishes.

$$\int_V (\nabla V_d)^{\,2} \, dV = 0$$

Integrating, $\quad (\nabla V_d)^{\,2} = 0 \quad \Rightarrow \quad \nabla V_d = 0$

$\Rightarrow \qquad\qquad V_d = V_1 - V_2$ = constant everywhere in V.

Hence, $V_d = 0$ or $V_1 = V_2$.

This is the uniqueness theorem: If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Theorem: Let u_1 and u_2 be harmonic functions with equal boundary values: $u_1 = u_2$ on $\partial\Omega$, where Ω is some bounded open set. Then $u_1 \equiv u_2$ in Ω.

SOLUTION: Suppose $\Delta u = 0$ in . Then putting $u = v$ of into the first Green's identity implies

$$\int_{\partial\Omega} \frac{\partial u}{\partial v} dS = \int_{\Omega} |\nabla u|^2 \, dx$$

Since that the latter integral is strictly positive unless is a constant. Set, then on the boundary of. So that the left hand side of the above integral identity is zero. It follows that in, hence. But on, hence The theorem is prove

6.6 *POISSON EQUATION:-*

The Poisson equation is a partial differential equation (PDE) that describes how a scalar field evolves over space. It is a special case of the more general Laplace equation, and it includes a source term. The Poisson equation is named after the French mathematician Siméon Denis Poisson, who made significant contributions to mathematical physics.

The one-dimensional Poisson equation is expressed as:

$$\frac{\partial^2 u}{\partial x^2} = f(x)$$

In two or more dimensions, it takes the form:

$$\nabla^2 u = f(x)$$

Here, $\nabla^2$is the Laplacian operator, u is the scalar field, $f(x)$ is the source term, x represent the spatial coordinates.

6.7 *TWO DIMENSIONAL LAPLACE EQUATION:-*

The Laplace equation in two dimensions is a partial differential equation that involves the second partial derivatives of a function. It is named after the French mathematician Pierre-Simon Laplace. In two dimensions, the Laplace equation is expressed as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Here, $u(x, y)$ is a function of two variables x and y, and the equation states that the sum of the second partial derivative of u with respect to x and the second partial derivative of u with respect to y is equal to zero.

6.8 *THREE DIMENSIONAL LAPLACE EQUATION:-*

The Laplace equation in three dimensions is a partial differential equation that describes a scalar field's behavior when there are no sources or sinks of the scalar quantity within the region of interest. In three dimensions, the Laplace equation is expressed as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Here, $u(x,y,z)$ is a function of three variables y, and z. The Laplace equation states that the sum of the second partial derivative of u with respect to x, the second partial derivative of u with respect to y, and the second partial derivative of u with respect to z is equal to zero.

SOLVED EXAMPLE

EXAMPLE1: Let u be a harmonic function in the interior of a rectangle $0 \le x \le a, 0 \le y \le b$ in the$xy -$ plane, satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \dots(1)$$

with
$$u(0,y)=0, u(a,y)=0 \quad \dots(2)$$

$$u(x,b)=0$$

and
$$u(x,0)=f(x)$$

The determine u for the above problem

SOLUTION: Suppose the equation (1) has the solution form

$$u(x,y)=X(x)\bar{Y}(y) \quad \dots(4)$$

Putting this value of u in (1), we have

$$X''Y+Y''X=0 \quad \text{or} \quad \frac{X''}{X}=-\frac{Y''}{Y} \quad \dots(5)$$

$\therefore x$ and y are independent, each side of (5), must equal to the same constant, say,μ, the we gat

$$X''-\mu X=0 \quad \dots(6)$$

$$Y'' + \mu Y = 0 \qquad ...(7)$$

Putting the value of (2) in (4), we have

$X(0)Y(0) = 0$ and $X(a)Y(y) = 0$ …(8)

$$X(0) \text{ and } X(a) = 0$$

Where we have taken $Y(y) \neq 0$, otherwise $u = 0$ which does no satisfy.

CaseI: Let $\mu = 0$, then from (6), we obtain

$$X(x) = Ax + B \qquad ...(9)$$

Using (8),(9) obtain $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) \equiv 0$.

CaseII: Suppose $= \lambda^2, \lambda \neq 0$. Then the equation(6) is

$$X(x) = Ae^{x\lambda} + Be^{-x\lambda} \qquad ...(10)$$

Using (8) and (9) gives, we have

$0 = A + B$ and $0 = Ae^{x\lambda} + Be^{-x\lambda}$ ...(11)

So $A = B = 0$ and $X(x) \neq 0$ and hence $u \neq 0$. so we reject $\mu = \lambda^2$.

CaseIII: Suppose $= -\lambda^2, \lambda \neq 0$. Then the equation(6) is

$$X(x) = Acos\lambda x + Bsin\lambda x \qquad ...(12)$$

Using (8), (12) gives $0 = A$ and $0 = Acos\lambda a + Bsin\lambda a$ so that

$A = 0$ and $sin\lambda a = 0$

Where we have taken $B \neq 0$, since otherwise $X(x) \neq 0$ and hence $u \neq 0$ which not satisfy.

Now $sin\lambda a = 0 \Rightarrow \lambda a = n\pi \Rightarrow \lambda = \frac{n\pi}{a}, n = 1,2,3, ...$

Hence

$$X_n(x) = B_n sin(n\pi x/a)$$

Using $\mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$, (7) reduces to

$$Y'' - -\left(\frac{n^2\pi^2}{a^2}\right)Y = 0$$

Whose solution is $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$

Using (3), gives $0 = X(x)Y(b)$ so that $Y(b) = 0$, where we obtain taken $X(x) \neq 0,$ since otherwise $u = 0$ which does not satisfy (3)

Again $Y(b) = 0 \;\; \Rightarrow \;\; Y_n(b) = 0$

Putting these values of C_n obtained by above equation, we have

$$Y_n(y) = D_n\left(e^{-n\pi y/a}e^{n\pi b/a} - e^{n\pi y/a}e^{-n\pi b/a}\right)/e^{n\pi b/a}$$

$$Y_n(y) = D_n\left(e^{-n\pi(b-y)/a} - e^{n\pi(b-y)/a}\right)/e^{n\pi b/a}$$

$$Y_n(y) = 2D_n \, sinh\left\{\frac{n\pi(b-y)}{a}\right\}, \quad as e^{\theta} - e^{-\theta} = 2sinh\theta$$

$\therefore \; U_n(x,y) = X_n(x)Y_n(y) = F_n sin\,(n\pi x/a)\, sinh\left\{\frac{n\pi(b-y)}{a}\right\}$ are the solution of (1), we consider more general solution, we have

$$U(x,y) = \sum_{n=1}^{\infty} F_n \, sin\,(n\pi x/a)\, sinh\left\{\frac{n\pi(b-y)}{a}\right\}$$

Putting $y = 0$ in above equation and using (3) and (4), we get

$$f(x) = \sum_{n=1}^{\infty}\{F_n sin\,(n\pi x/a)\}\, sin\,(n\pi x/a)$$

Hence, we have

$$F_n \, sin\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\frac{n\pi x}{a}dx$$

$F_n = \frac{2}{a\,sin\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\frac{n\pi x}{a}dx$.is required solution.

EXAMPLE2: Solve two dimensional Laplace's equation in plane polar coordinates $r(r,\theta)$

SOLUTION: Let the Laplace's equation in $\mathbb{R}^2$ in terms of the usual (i.e., Cartesian) (x,y) coordinate system is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0. \tag{1}$$

The Cartesian coordinates can be described by the polar coordinates as follows:

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta. \end{cases} \tag{2}$$

Let the first partial derivatives of x, y $ie., r, \theta$, we get

$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta, & \dfrac{\partial x}{\partial \theta} = -r\sin\theta; \\ \dfrac{\partial y}{\partial r} = \sin\theta, & \dfrac{\partial y}{\partial \theta} = r\cos\theta. \end{cases} \tag{3}$$

Now we will use Chain Rule since (x, y) the function of (r, θ), as expressed as in (2)

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{using (3)} \\ &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}. \end{aligned} \tag{4}$$

Again differentiating, we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\ &= \cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) \\ &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}. \end{aligned} \tag{5}$$

Similarly

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\ &= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}. \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial y} \\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right) - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial x \partial y} r\cos\theta\right) \\
&\quad - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial^2 u}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2} r\cos\theta\right) \\
&= -r\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}$$

Dividing both sides with r^2, using (4)

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \qquad (6)$$

Finally adding (4) and (5), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Hence the Laplace's equation (1) occur that

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Hence, this is obtained the Laplace's equation in the polar coordinate system.

EXAMPLE3: Solve two dimensional Laplace's equation in cylindrical coordinates (r, θ, z)

SOLUTION: Laplace's equation in three dimensional Cartesian coordinates (x, y, z) is

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \;\; ie., \qquad \psi_{xx} + \psi_{yy} + \psi_{zz} = 0 \ldots (1)$$

And the relations between Cartesian and Cylindrical coordinates are

$$x = rcos\theta, y = rsin\theta, z = z, i.e., r^2 = x^2 + y^2, \theta = \tan^{-1}\frac{y}{x}, z = z$$

Now $\psi_x = \psi_r r_x + \psi_\theta \theta_x + \psi_z z_x = \psi_r cos\theta - \psi_\theta . \frac{sin\theta}{r}$

$$\psi_y = \psi_r r_y + \psi_\theta \theta_y + \psi_z z_y = \psi_r sin\theta + \psi_\theta . \frac{cos\theta}{r}$$

$$\psi_z = \psi_r r_z + \psi_\theta \theta_z + \psi_z z_z = \psi_z$$

Also, $\psi_{xx} = (\psi_x)_r r_x + (\psi_x)_\theta \theta_x + (\psi_x)_z z_x$

$$= (\psi_x)_r r_x + (\psi_x)_\theta \theta_x + (\psi_x)_z z_x$$

$$= \left(\psi_r cos\theta - \psi_\theta \frac{sin\theta}{r}\right)_r cos\theta - \left(\psi_r cos\theta - \psi_\theta \frac{sin\theta}{r}\right)_r \frac{sin\theta}{r}$$

$$= \psi_{rr} cos^2\theta - \psi_{\theta r}\frac{sin\theta cos\theta}{r} + \psi_\theta \frac{sin\theta cos\theta}{r^2} - \psi_{r\theta}\frac{sin\theta cos\theta}{r} + \psi_r \frac{sin^2\theta}{r} + \psi_{\theta\theta}\frac{sin^2\theta}{r^2} + \psi_\theta \frac{sin\theta cos\theta}{r^2}$$

$$\psi_{xx} = \psi_{rr} cos^2\theta - 2\psi_{r\theta}\frac{sin\theta cos\theta}{r} + \psi_{\theta\theta}\frac{sin^2\theta}{r^2} + \psi_r \frac{sin^2\theta}{r} + 2\psi_\theta \frac{sin\theta cos\theta}{r^2}$$

Similarly

$$\psi_{yy} = \psi_{rr} sin^2\theta + 2\psi_{r\theta}\frac{sin\theta cos\theta}{r} + \psi_{\theta\theta}\frac{cos^2\theta}{r^2} + \psi_r \frac{sin^2\theta}{r} + 2\psi_\theta \frac{sin\theta cos\theta}{r^2}$$

and $$\psi_{zz} = \psi_{zz}$$

Putting these values of ψ_{xx}, ψ_{yy} and ψ_{zz} in (1), we given the Laplace's equation in cylindrical coordinates (r, θ, z) as

$\psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} + \frac{1}{r}\psi_r + \psi_{zz} = 0$, ie., $\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = 0$

EXAMPLE4: Solve two dimensional Laplace's equation in spherical polar coordinates (r, θ, ϕ).

SOLUTION: The relations between Cartesian and Cylindrical coordinates are

$$x = rsin\theta cos\phi, \qquad y = rsin\theta sin\phi, \qquad z = rcos\theta, \qquad i.e.,$$

$$r^2 = x^2 + y^2, cos\theta = \frac{z}{r}, tan\phi = \frac{y}{x}$$

Since $$r_x = \frac{x}{r}, r_y = \frac{y}{r}, r_z = \frac{z}{r}$$

$$\theta_x = \frac{cos\theta cos\phi}{r}, \theta_y = \frac{cos\theta sin\phi}{r}, \theta_x = -\frac{sin\theta}{r}$$

$$\phi_x = -\frac{sin\phi}{rsin\theta}, \phi_y = -\frac{cos\phi}{rsin\theta}, \phi_z = 0$$

Now

$$\psi_x = \psi_r r_x + \psi_\theta \theta_x + \psi_\phi \phi_x = \psi_r sin\theta cos\phi + \psi_\theta . \frac{cos\theta cos\phi}{r} - \psi_\phi \frac{sin\phi}{rsin\theta}$$

$$\psi_y = \psi_r r_y + \psi_\theta \theta_y + \psi_z z_y = \psi_r sin\theta sin\phi + \psi_\theta . \frac{cos\theta sin\phi}{r} + \psi_\phi \frac{cos\phi}{rsin\theta}$$

$$\psi_z = \psi_r r_z + \psi_\phi \theta_z + \psi_\phi \phi_z = \psi_r cos\theta - \psi_\theta \frac{sin\theta}{r}$$

Also

$$\psi_x = \psi_r r_x + \psi_\theta \theta_x + \psi_\phi \phi_x$$

$$= \left(\psi_r sin\theta \, cos\phi + \psi_\theta \frac{cos\theta cos\phi}{r} - \psi_\phi \frac{sin\phi}{rsin\theta}\right)_r sin\theta cos\phi$$

$$+ \left(\psi_r sin\theta \, cos\phi + \psi_\theta \frac{cos\theta cos\phi}{r} - \psi_\phi \frac{sin\phi}{rsin\theta}\right)_\theta \frac{cos\theta cos\phi}{r}$$

$$+ \left(\psi_r sin\theta \, cos\phi + \psi_\theta \frac{cos\theta cos\phi}{r} - \psi_\phi \frac{sin\phi}{rsin\theta}\right)_\phi \left(-\frac{sin\phi}{rsin\theta}\right)$$

$$= \left(\psi_{rr} sin\theta cos\phi + \psi_{\theta r}\frac{cos\theta cos\phi}{r} - \psi_{\theta}\frac{cos\theta cos\phi}{r} - \psi_{\phi r}\frac{sin\phi}{rsin\theta} + \psi_{\phi}\frac{sin\phi}{r^2 sin\theta}\right) sin\theta cos\phi$$
$$+ \left(\psi_{r\theta} sin\theta cos\phi + \psi_r cos\theta cos\phi + \psi_{\theta\theta}\frac{cos\theta cos\phi}{r} - \psi_{\phi}\frac{sin\theta cos\phi}{r} - \psi_{\phi\theta}\frac{sin\phi}{rsin\theta} + \psi_{\phi}\frac{cos\theta sin\phi}{rsin^2\theta}\right)\frac{cos\theta cos\phi}{r}$$
$$- \left(\psi_{r\phi} sin\theta cos\phi - \psi_r sin\theta sin\phi + \psi_{\theta\phi}\frac{cos\theta cos\phi}{r} - \psi_{\theta}\frac{cos\theta sin\phi}{r} - \psi_{\phi\phi}\frac{sin\phi}{rsin\theta} - \psi_{\phi}\frac{cos\phi}{rsin\phi}\right)\frac{sin\phi}{rsin\theta}\psi_{xx}$$
$$= \psi_{rr} sin^2\theta cos^2\phi + \psi_{\theta\theta}\frac{cos^2\theta cos^2\phi}{r^2} + \psi_{\phi\phi}\frac{sin^2\phi}{r^2 sin^2\theta} + 2\psi_{\theta r}\frac{sin\theta cos\theta cos^2\phi}{r} - 2\psi_{r\phi}\frac{sin\theta cos\phi}{r} - 2\psi_{\theta\phi}\frac{sin\theta cos\theta cos\phi}{r^2} + \psi_r\left(\frac{cos^2\theta cos^2\phi}{r} + \frac{sin^2\phi}{r}\right)$$
$$+ \psi_{\theta}\left(\frac{cos\theta sin^2\psi}{r^2 sin\theta} - \frac{sin\theta cos\theta cos^2\phi}{r^2}\right) + \psi_{\phi}\left(\frac{sin\phi cos\phi}{r^2} + \frac{sin\phi cos\phi cos^2\theta}{r^2 sin^2\theta} + \frac{sin\phi cos\phi}{r^2 sin^2\theta}\right)$$

Similarly, $\psi_{yy} = \psi_{rr} sin^2\theta sin^2\phi + \psi_{\theta\theta}\frac{cos^2\theta sin^2\phi}{r} + \psi_{\phi\phi}\frac{cos^2\phi}{r^2 sin^2\theta} + 2\psi_{r\theta}\frac{sin\theta cos\theta sin^2\phi}{r^2} + 2\psi_{r\phi}\frac{sin\phi cos\phi}{r} + 2\psi_{\theta\phi}\frac{sin\phi cos\theta cos\phi}{r^2} + \psi_r\left(\frac{cos^2\theta cos^2\phi}{r} + \frac{cos^2\phi}{r}\right) + \psi_{\theta}\left(\frac{cos\theta cos^2\phi}{r^2 sin\theta} - \frac{2sin\theta sin\theta sin^2\phi}{r^2}\right) - \psi_{\phi}\left(\frac{sin\phi cos\phi}{r^2} + \frac{sin\phi cos\phi cos^2\theta}{r^2 sin^2\theta} + \frac{sin\phi cos\phi}{r^2 sin^2\theta}\right)$

$$\psi_{zz} = \psi_{rr} cos^2\theta + \psi_{\theta\theta}\frac{sin^2\phi}{r} - 2\psi_{r\theta}\frac{sin\theta cos\theta}{r} + \psi_r\frac{sin^2\phi}{r} + \psi_{\theta}\frac{sin\theta cos\theta}{r}$$

Putting these values of ψ_{xx}, ψ_{yy} and ψ_{zz} in (1), we given the Laplace's equation in cylindrical coordinates (r, θ, ϕ) as

$\nabla^2\psi = \psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} + \frac{1}{r^2 sin^2\theta}\psi_{\phi\phi} + \frac{2}{r}\psi_r + \frac{cos\theta}{r^2 sin^2\theta}\psi_{\theta} = 0, i.e.,$

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2\psi}{\partial\psi^2}=0$$

EXAMPLE5: Obtain a solution of Laplace's equation in rectangular Cartesian coordinates (x, y, z) by the method of separation of variables.

SOLUTION: let we know that three dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad \dots(1)$$

From (1), we have

$$u(x,y,z)=X(x)Y(y)Z(z) \quad \dots(2)$$

where X, Y and Z are functions of x, y and z respectively.

Putting the value of u in (1), we get

$$\frac{X''}{X}+\frac{Y''}{Y}=-\frac{Z''}{Z} \qquad \dots(3)$$

Since x, y and z are independent variables.

The following three cases arise:

CaseI: If equation (3) zero, then

$$X''=0, Y''=0, Z''=0$$

giving,

$$X=Ax+B, Y=Cy+D, Z=Ez+F$$

Hence

$$u(x,y,z)=(Ax+B)(Cy+D)(Ez+F)$$

CaseII: Let $\frac{X''}{X}=\lambda_1{}^2, \frac{Y''}{Y}=\lambda_2^2$ and $\lambda_1^2+\lambda_2^2=\lambda^2$. Then (3) give

$$X''-\lambda_1^2X=0, Y''-\lambda_1^2Y=0, Z''+\lambda^2Z=0$$

$\therefore \quad X=Ae^{x\lambda_1}+Be^{-x\lambda_1}; \quad Y=Ce^{y\lambda_2}+De^{-x\lambda_2};$ and $Ecos\lambda z+Fsin\lambda z$

Hence

$$u(x,y,z)=\left(Ae^{x\lambda_1}+Be^{-x\lambda_1}\right)\left(Ce^{y\lambda_2}+De^{-x\lambda_2}\right)(Ecos\lambda z+Fsin\lambda z)$$

A more general solution is obtained by

$$u(x,y,z)=\sum_{\lambda_1}\sum_{\lambda_2}\left(Ae^{x\lambda_1}+Be^{-x\lambda_1}\right)\left(Ce^{y\lambda_2}+De^{-x\lambda_2}\right)(Ecos\lambda z+Fsin\lambda z)$$

CaseIII: Let $\frac{X''}{X}=-\lambda_1{}^2, \frac{Y''}{Y}=-\lambda_2^2$ and $-(\lambda_1^2+\lambda_2^2)=-\lambda^2$, then

$$X''+\lambda_1^2X=0, Y''+\lambda_1^2Y=0, Z''-\lambda^2Z=0$$

$$X=A\cos\lambda_1x+Bsin\lambda_1x, \quad y=Ccos\lambda_2y+Dsin\lambda_2y$$

And $\quad Z=Ee^{z\lambda}+Fe^{-z\lambda}$

Hence the general solution is written as

$$u(x,y,z) = \sum_{\lambda_1}\sum_{\lambda_2}(A\cos\lambda_1 x + Bsin\lambda_1 x\ )\,(Ccos\lambda_2 y + Dsin\lambda_2 y)\big(Ee^{z\lambda} + Fe^{-z\lambda}\big)$$

SELF CHECK QUESTIONS

1. What is Laplace's equation, and what does it represent in terms of a scalar field?
2. Explain the physical interpretation of Laplace's equation. In what types of situations does it commonly arise?
3. How is Laplace's equation written in three dimensions and what is the Laplacian operator denoted by?
4. Why are boundary conditions necessary when solving Laplace's equation? Provide examples of types of boundary conditions.
5. How does Poisson's equation differ from Laplace's equation?

6.9 *SUMMARY:-*

In this unit we have studied the Laplace and Poisson equations is crucial in various scientific and engineering fields, providing mathematical tools for analyzing and predicting the behavior of scalar fields in different physical situations.

Overall Laplace's equation describes situations where a scalar field is in a state of equilibrium with no sources or sinks, while Poisson's equation includes a source term and is used when there are localized sources or sinks in the field. Both equations are fundamental in physics and engineering for modeling various phenomena related to heat, potential fields, and fluid flow.

6. 10 *GLOSSARY:-*

- **Laplace Equation:** A partial differential equation stating that the Laplacian of a scalar field is zero.

$$\nabla^2 u = 0$$

- **Laplacian:** A mathematical operator representing the divergence of the gradient. In Cartesian coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- **Equilibrium:** A state in which a system is not changing with respect to time, described by the Laplace equation.
- **Scalar Field:** A function that assigns a scalar value to every point in space.
- **Poisson Equation:** A partial differential equation similar to Laplace's equation but with a source term.

$$\nabla^2 u = f$$

  Where f is the short term.
- **Source Term:** The term f in the Poisson equation representing an external influence or source that affects the scalar field u.
- **External Influence:** Factors from outside the system that affect the field in the Poisson equation, such as charge density or external forces.
- **Distributed Source:** A source term f that varies across space, representing a distribution of sources.
- **Boundary Conditions:** Conditions imposed on the solution of a differential equation to determine a unique solution. Crucial in solving both Laplace and Poisson equations.
- **Steady-State:** A condition where a system's properties do not change with time. Laplace equation describes such states.
- **Equilibrium Solution:** The solution to the Laplace equation representing a stable, unchanging state.
- **Numerical Methods:** Techniques for approximating solutions to differential equations, often used for solving Laplace and Poisson equations in complex geometries.

Understanding these terms is essential for working with Laplace and Poisson equations, as they form the foundation for studying scalar fields and their behavior in various physical and engineering applications.

*6.*11 *REFERENCES:-*

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- Walter A. Strauss (2008), Partial Differential Equations: An Introduction" by Walter A. Strauss:

6.12 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- https://epgp.inflibnet.ac.in/Home/ViewSubject?catid=ZLCHeZEhCZ8yCri36nSF3A==
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6.13 *TERMINAL QUESTION:-*

(TQ-1): Find the potential function $\psi(x, y, z)$ in the region

$0 \le x \le a, 0 \le y \le b, 0z \le c$ satisfying the conditions.

i. $\psi = 0$, on $x = 0, x = a, y = 0, y = b, z = 0$.

ii. $\psi = f(x, y)$ on $z = c, 0 \le x \le a, 0 \le y \le b$

(TQ-2): State Laplace equation in Cartesian coordinates.

(TQ-3): Obtain the general solution of Laplace's equation in cylindrical coordinates for the case of axial symmetry, namely

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0$$

(TQ-4): Obtain a solution of Laplace's equation in rectangular Cartesian coordinates (x, y, z) by the method of separation of variables.

(TQ-5): Let u_1 andu_2 be harmonic functions with equal boundary values: $u_1 = u_2$on $\partial\Omega$, where Ω is some bounded open set, then prove that $u_1 \equiv u_2$in Ω.

6.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. Laplace's equation is $\Delta\phi = 0$, representing a second-order partial differential equation that describes a scalar field ϕ where the Laplacian of ϕ is equal to zero. It represents situations where the field is in a steady state with no sources or sinks.
2. Laplace's equation commonly arises in physics and engineering to describe steady-state conditions where there are no sources or sinks. It can model phenomena such as heat conduction, electric potential, and fluid flow in equilibrium.
3. In three dimensions, Laplace's equation is written as $\nabla^2\phi = 0$, where ∇^2 is the Laplacian operator, also known as del squared.

4. Boundary conditions are necessary to obtain a unique solution to Laplace's equation. Examples include Dirichlet boundary conditions (specifying function values on the boundary) and Neumann boundary conditions (specifying the normal derivative on the boundary).
5. Poisson's equation is $\Delta\phi = f$, where Δ is the Laplacian operator, ϕ is the scalar field, and f is a given source term. Unlike Laplace's equation, Poisson's equation accounts for localized sources or sinks in the field.

TERMINAL ANSWERS

(TQ-1):

i. $\psi(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} G_{mn}\, sinh(\lambda_{mn}c) sin\frac{m\pi x}{a}\ sin\frac{n\pi y}{b} \sin(\lambda_{mn}z),$

ii. $G_{mn} = \frac{4cosech(\lambda_{mn}z)}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x, y)\ sin\frac{m\pi x}{a}\ sin\frac{n\pi y}{b} dxdy$

Unit 7: Dirichlet's Problem and Newmann Problem for a Rectangular

CONTENTS:

7.1 *INTRODUCTION:-*

The solution of the two-dimensional Laplace equation involves finding a scalar field u that satisfies the equation =0 within a specified region. This equation describes the steady-state distribution of a scalar quantity in two dimensions. The process includes defining the problem's region, expressing the Laplace equation in Cartesian or other coordinate systems, specifying boundary conditions, choosing a solution technique (such as separation of variables or numerical methods), solving for u, and ensuring the consistency of the solution by checking both the Laplace equation and the specified boundary conditions. The obtained solution provides insights into the scalar field's behavior within the region, aiding in the understanding of physical or mathematical systems.

The solution to the two-dimensional Laplace equation is often expressed as a mathematical function or series that describes the behavior of the scalar field u throughout the defined region. The specific solution technique used will depend on the nature of the problem and the given boundary conditions.

7.2 *OBJECTIVES:-*

After studying this unit, you will be able to the Dirichlet and Neumann problems are boundary value problems in the context of partial differential equations (PDEs) that typically arise in the study of physical phenomena such as heat conduction, fluid flow, and electrostatics. These problems are often discussed in the context of rectangular domains. Let's look at the objectives of the Dirichlet and Neumann problems for a rectangular domain:

- Given a rectangular region in space and a partial differential equation [such as the Laplace equation or the Poisson equation(we have already studied in previous unit)], the Dirichlet problem seeks to find a solution that satisfies the PDE within the region while specifying the values of the solution on the boundary of the region.

- Similar to the Dirichlet problem, the Neumann problem seeks a solution to a PDE within a rectangular region. However, instead of prescribing the function values on the boundary, the Neumann problem prescribes the normal derivative (flux) of the solution on the boundary.

7.3 *LAPLACE'S EQUATION IN CARTESIAN COORDINATS*(x, y)*:-*

Laplace's equation in two dimensions using the method of separation of variables. Three coordinate systems are considered for the analysis: (i) Cartesian Coordinates, (ii) Polar Coordinates, and (iii) Spherically Polar Coordinates. The previous unit we have already studied the Laplace two dimensional in Polar Coordinates, and (iii) Spherically Polar Coordinates and cylindrical coordinates. In this unit we will study the Cartesian form of Laplace equation.

Now Let us consider the solution of Laplace equation is given below

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \dots (1)$$

Since $$u(x, y) = X(x)Y(y) \qquad \dots (2)$$

Putting the value in (1), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2u}{dx^2} = k \qquad \dots (3)$$

where k is known as the separation constant.

CaseI: Let $k = p^2 > 0$, p being real, from (3), we have obtain

$$\frac{d^2X}{dx^2} - p^2X = 0, \frac{d^2X}{dx^2} + p^2Y = 0$$

Whose solutions are respectively

$$X(x) = c_1 e^{px} + c_2 e^{-px}, Y(y) = c_3 cospy + c_4 sinpy$$

Hence the equation (1) is obtain

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 cospy + c_4 sinpy) \quad \dots (4)$$

Where c_j $(j = 1,2,3,4)$ are constants.

CaseII: Let $k = 0, p$ being real, from (3), we can be written as

$$X(x) = c_5 x + c_6, Y(y) = c_7 y + c_8$$

So that

$$u(x, y) = (c_5 x + c_6)(c_7 y + c_8) \quad \dots (5)$$

Where c_j $(j = 5,6,7,8)$ are constants.

CaseIII: Let $k = -p^2 < 0,$ then the case I , the solution of the equation (1) is

$$u(x, y) = (c_9 cospy + c_{10} sinpy)(c_{11} e^{px} + c_{12} e^{-px}) \quad \dots (6)$$

where c_j $(j = 9,10,11,12)$ are constants.

7.4 *DIRICHLET PROBLEM FOR A RECTANGULE:-*

The Dirichlet problem is a classical problem in partial differential equations that seeks to find a solution to the Laplace's equation within a given domain, subject to specified boundary conditions.

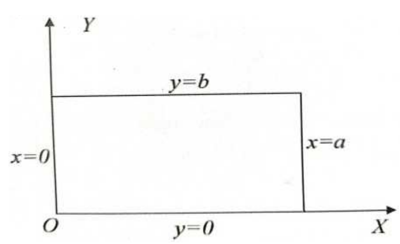

Fig.1

Interior Dirichlet's problem for a rectangle is explained as below:

To obtain the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at any interior point of the rectangle $0 \le x \le a, 0 \le y \le b$ subject to boundary condition

$$u(0, y) = u(a, y) = u(x, b) = 0, \quad u(x, 0) = f(x)$$

Where $f(x)$ is supposed to be expansible in Fourier sine series.

Let us assume the equation(4) is

$$c_1e^{pa} + c_2e^{-pa}, c_1 + c_2 = 0 \quad (\because c_3cospy + c_4sinpy \neq 0)$$

Writing $c_1 = c_2 = 0$ so that $u(x, b) = 0$, is only trivial solution.
Similarly, the equation (5),it only trivial solution $u(x, y) = 0$.

Using the B.C.(Boundary Condition) $u(0, y) = u(a, y) = 0$, now from (6) that $c_9 = 0$ and

$$c_{10}sinpa(c_{11}e^{py} + c_{12}e^{-py}) = 0$$

Since $c_{10} \neq 0 \Rightarrow sinpa = 0, i.e., p = \frac{n\pi}{a}, (n = 1,2, \ldots.)$
Thus, we obtain

$$u(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)[a_n e^{n\pi y/a} + b_n e^{-n\pi y/a}]$$

So $u(x, b) = 0$, then

$$a_n e^{n\pi b/a} + b_n e^{-n\pi b/a} = 0 \Rightarrow b_n = \frac{a_n}{e^{-n\pi b/a}} e^{n\pi b/a}$$

$$u(x, y) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\left[\exp\left\{\frac{n\pi(y-b)}{a}\right\} - \exp\left\{-\frac{n\pi(y-b)}{a}\right\}\right]$$

Where $A_n = 2a_n exp\left(\frac{n\pi x}{a}\right)$. Now we can write

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi(y-b)}{a}\right\}$$

Hence the non homogeneous boundary condition $u(x, 0) = f(x)$ obtain

$f(x) = \sum_{n=1}^{\infty} A_n \, sin\left(\frac{n\pi x}{a}\right) sinh\left\{-\frac{n\pi b}{a}\right\}$ *is half range Fourier series.*

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi(y-b)}{a}\right\} \quad \ldots(7)$$

where $A_n = -\frac{2}{a\, sinh\left\{\frac{n\pi b}{a}\right\}}\int_0^a f(x)\, sin\left(\frac{n\pi x}{a}\right) dx \quad \ldots(8)$

7.5 *NEUMANN PROBLEM FOR A RECTANGULE:-*

The Neumann problem for a rectangle involves finding a solution to Laplace's equation inside the rectangle while prescribing the values of the derivative of the solution (usually the normal derivative) on parts of the rectangle's boundary.

Interior Neumann problem for a rectangle is explained as below:

To obtain the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at any interior point of the rectangle $0 \leq x \leq a, 0 \leq y \leq b$ subject to boundary condition

$$\frac{\partial u(0,y)}{\partial x} = \frac{\partial u(a,y)}{\partial x} = \frac{\partial u(x,0)}{\partial x} = 0, \frac{\partial u(x,b)}{\partial y} = f(x)$$

where $f(x)$ is considered to be effective in Fourier cosine series.

Let us suppose the equation (4) can be written as

$$u(x,y) = (c_3 e^{py} + c_4 e^{-pxy})(c_1 cospx + c_2 sinpx)$$

Since the boundary condition $\frac{\partial u(0,y)}{\partial x} = 0$, shows that $c_2 = 0$ and the condition $\frac{\partial u(a,y)}{\partial x} = 0 \Rightarrow sinpa = 0 i.e., p = \frac{n\pi}{a}, (n = 0,1,2, \ldots)$.

Also $\frac{\partial u(x,0)}{\partial x} = 0$ obtains $c_4 = c_3$.

Hence the equation can be obtained from the above equations

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) \cosh\left\{\frac{n\pi y}{a}\right\}$$

$f(x) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{a} cos\left(\frac{n\pi x}{a}\right) sinh\left\{\frac{n\pi b}{a}\right\}$ is half Fourier series, then

$$A_n \frac{n\pi}{a} sinh\left\{\frac{n\pi b}{a}\right\} = \frac{2}{a}\int_0^a f(x)\, cos\left(\frac{n\pi x}{a}\right) dx$$

Finally the solution of interior Neumann problem is obtained by

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \frac{n\pi}{a} cos\left(\frac{n\pi x}{a}\right) sinh\left\{\frac{n\pi b}{a}\right\}$$

Where A_0 is arbitrary constant and

$$A_n = \frac{2}{n\pi\, sinh\left\{\frac{n\pi b}{a}\right\}} \int_0^a f(x)\, cos\left(\frac{n\pi x}{a}\right) dx$$

SOLVED EXAMPLE

EXAMPLE1: Find the steady state temperature distribution in a rectangular plate of sides a and b insulted at the lateral surface and satisfying the boundary conditions.

$$u(0,y) = u(a,y) = 0 \; for \; 0 \leq y \leq b$$

and $u(x,b) = 0$ and $u(x,0) = x(a-x) \; for \; 0 \leq x \leq a$

SOLUTION: First proceed in equation (8), we get

The present problem

$$u(x,0) = f(x) = x(a-x), 0 \leq x \leq a$$

$$A_n = \frac{2}{a \, sinh\left\{\frac{n\pi b}{a}\right\}} \int_0^a f(x) \, sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a \, sinh\left\{\frac{n\pi b}{a}\right\}} \int_0^a (ax - x^2) \, sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a \, sinh\left\{\frac{n\pi b}{a}\right\}} \Bigg[(ax - x^2)\left(-\frac{a}{n\pi}\right) \cos\frac{n\pi b}{a} - (a - 2x)\left(-\frac{a^2}{n^2\pi^2}\right) \sin\frac{n\pi x}{a} + (-2)\left(-\frac{a^3}{n^3\pi^3}\right) cos\frac{n\pi x}{a}\Bigg]_0^a$$

$$= \frac{2}{a \, sinh\left\{\frac{n\pi b}{a}\right\}} \left[-\frac{2a^3(-1)^n}{n^3\pi^3} + \frac{2a^3}{n^3\pi^3}\right] = \frac{4a^3}{n^3\pi^3} [1 - (-1)^n] \, cosec\frac{n\pi b}{a}$$

$$= \begin{cases} 0, if \; n = 2m(even) and \; m = 1,2,3, \ldots. \\ \left\{\frac{8a^2}{(2m-1)^3\pi^3}\right\} cosech\left\{\frac{(2m-1)\pi b}{a}\right\}, if \; n = 2m - 1 \; and \; m = 1,2,3 \ldots. \end{cases}$$

Putting the above values in equation (8), we obtain

$$u(x,y) = \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^3} \sinh\frac{(2m-1)(b-y)\pi}{a} cosech\frac{(2m-1)\pi b}{a}$$

EXAMPLE2: Solve the Poisson's equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2, \qquad 0\le x\le 1, 0\le y\le 1$$

with the boundary condition $\psi = 0$ on sides $x = 0,1, y = 0,1$.

SOLUTION: Let us consider the poisson's equation of the form $\psi(x,y) = f(x,y) + g(x,y)$ where $f(x,y)$ is the solution of Laplace's equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ and $g(x,y)$ is a particular solution $\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=2$. It is customary to assume g in the form

$$g = ABx + Cy + Dx^2 + Exy + Fy^2,$$

where A,B,C,D,E,F are constants. Then

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=2 \quad lead\ to\ D+F=1$$

Now we take $D = 1, F = 0$.

Therefore,

$$g(x,y) = -x + x^2$$

So $g = 0 \ \Rightarrow x = 0,1$.

Let $f(x,y)$ form the equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0, \qquad \ldots(1) \qquad 0\le x\le 1, 0\le y\le 1$$

Satisfying

$$f(0,y) = -g(0,y) = 0, \qquad f(1,y) = -g(1,y) = 0$$

$$f(x,0) = -g(x,0) = x - x^2, \qquad f(x,1) = -g(x,1) = x - x^2$$

Using the method of separation of variables, the $f(x,y)$ is

$$f(x,y) = (c_1e^{py} + c_2e^{-py})(c_3cospx + c_4sinpx)$$

p^2 being constant. Since $f(0,y) = 0$ obtain $c_3 = 0$, while the condition $f(1,y) = 0 \ \Rightarrow sinp = 0ie., p = n\pi, (n = 1,2,3 \ldots)$.

Then according to superposition principle, we obtain

$$f(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\,(A_ne^{n\pi y} + B_ne^{-n\pi y}) \quad \ldots(1)$$

Now we use non-homogeneous boundary condition $f(x,0) = x - x^2$ obtain

$$x - x^2 = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

Where $a_n = A_n + B_n$ and $f(x,1) = x - x^2$ show to

$$x - x^2 = \sum_{n=1}^{\infty} \sin(n\pi x)\{a_n \cosh(n\pi) + b_n sinh(n\pi)\}$$

Where $b_n = A_n - B_n$

Since then follows that this condition

$$a_n = 2\int_0^1 (x - x^2)\, sin(n\pi x)dx = \frac{4}{n^3\pi^3}[1 - (-1)^n]$$
$$= \begin{cases} 0, & if\ n\ is\ even \\ \dfrac{8}{n^3\pi^3}, & if\ n\ is\ odd \end{cases}$$

and

$$\{a_n \cosh(n\pi) + b_n sinh(n\pi)\} = 2\int_0^1 (x - x^2)\, sin(n\pi x)dx = a_n$$

So that $b_n = \frac{a_n(1 - cosh(n\pi))}{sinh(n\pi)}$.

Putting the value of a_n and b_n in (1), we obtain

$$f(x,y) = \sum_{n=1}^{\infty} \frac{a_n sin(n\pi x)}{sinh(n\pi)}\ \{a_n\, sinh[(1-y)(n\pi)] + b_n sinh(n\pi y)\}$$

Thus the Poisson's equation is

$$\psi(x,y) = x - x^2 + \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{a_n \sin\{(2n-1)\pi x\}}{(2n-1)^3 sinh\{(2n-1)\pi\}}\ [sinh[(2n-1)(1-y)\pi] + b_n \sinh\{(2n-1)\pi y\}]$$

EXAMPLE3: A thin rectangular homogeneous thermally conducted plate occupies the region $0 \le x \le a, 0 \le y \le b$. The edge $y = 0$ is held at temperature $Tx(x - a)$, where T is constant and the other edges are maintained at 0^0. The other faces are insulted and there is no source or sink inside the plate. Find the steady state temperature inside the plate.

SOLUTION: Let the given Laplace's equation is

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0, \qquad \dots(1)$$

and subject to the given conditions

$$f(0,y)=-f(a,y)=f(x,b)=0, \qquad f(x,0)=Tx(x-a)$$

From the equation (7), we put $g(x)=T(x-a)$, get

$$f(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi(y-b)}{a}\right\}$$

Where

$$A_n=-\frac{2}{a\,sinh\left\{\frac{n\pi b}{a}\right\}}\int_0^a f(x)\,sin\left(\frac{n\pi x}{a}\right)d$$

$$=-\frac{2T}{a\,sinh\left\{\frac{n\pi b}{a}\right\}}\int_0^a x(x-a)\,sin\left(\frac{n\pi x}{a}\right)dx$$

$$=-\frac{4Ta^2\{(-1)^n-1\}}{n^3\pi^3 sinh\left(\frac{n\pi x}{a}\right)}=\begin{cases}0, & if\ n\ is\ even\\ \dfrac{8}{n^3\pi^3 sinh\left(\frac{n\pi x}{a}\right)}, & if\ n\ is\ odd\end{cases}$$

Hence

$f(x,y)$

$$=\frac{8Ta^2}{\pi^3}\sum_{n=1}^{\infty}\frac{a_n\,\mathrm{cosech}\left\{\frac{(2n-1)\pi b}{a}\right\}}{(2n+1)^3}sin\left\{\frac{(2n+1)\pi x}{a}\right\}sinh\left\{\frac{(2n+1)(y-b)\pi}{a}\right\}$$

SELF CHECK QUESTIONS

1. What is Dirichlet's Problem for a rectangle?
2. How are the boundary conditions specified in Dirichlet's Problem for a rectangle?
3. What type of physical problems can be modeled using Dirichlet's Problem for a rectangle?
4. What is Neumann's Problem for a rectangle?
5. How are the boundary conditions specified in Neumann's Problem for a rectangle?
6. Provide an example where Neumann's Problem for a rectangle is relevant in a real-world scenario.
7. Can Dirichlet's Problem for a rectangle be applied to model electrical potential distribution?
8. How are Dirichlet's and Neumann's Problems related to partial differential equations?
9. Question: In the context of Dirichlet's Problem, what is a harmonic function?

7.6 *SUMMARY:-*

In this unit we have studied the concept in Dirichlet's problem and Neumann's problem of the mathematical concept in potential theory. In which Dirichlet's problem is involves finding a harmonic function within a specified region (in this case, a rectangle) with prescribed boundary values. In simpler terms, it seeks a solution to Laplace's equation inside the rectangle, subject to given values on its boundary. Neumann's problem is also a boundary value problem in potential theory, but in this case, it deals with finding a harmonic function within a specified region (rectangle) such that its normal derivative on the boundary is equal to given values. In other words, instead of specifying the function values on the boundary, Neumann's problem specifies the derivative of the function with respect to the normal direction on the boundary.

7.7 *GLOSSARY:-*

- **Harmonic Function:** A function that satisfies Laplace's equation, indicating that the sum of its second partial derivatives with respect to each variable is zero.
- **Normal Derivative:** The derivative of a function in the direction perpendicular to the boundary. In Neumann's Problem, the normal derivative of the harmonic function is specified on the boundary of the rectangle.
- **Potential Theory:** A branch of mathematical analysis dealing with the study of harmonic functions and their applications, often in the context of gravitational and electric fields.
- **Laplace's Equation:** A partial differential equation satisfied by harmonic functions, expressed as the sum of the second partial derivatives of the function with respect to each variable being equal to zero.
- **Boundary Value Problem:** A mathematical problem where the values of a solution or its derivatives are specified on the boundary of a given region.
- **Neumann Boundary Conditions:** Conditions specified on the boundary of a region for Neumann's Problem, indicating the normal derivative of the harmonic function.

- **Rectangular Domain:** The specified region or area where Neumann's Problem is considered, typically defined by a rectangle in two-dimensional space.
- **Mathematical Modeling:** The process of translating real-world problems into mathematical terms, essential in formulating and solving problems such as Neumann's Problem.
- **Dirichlet Boundary Conditions:** Conditions specified on the boundary of a region for Dirichlet's Problem, indicating the values that the harmonic function should take on the boundary of the rectangle.
- **Homogeneous Equation:** An equation in which the sum of its terms is zero. Laplace's equation is a homogeneous PDE because it equates the Laplacian of a function to zero.
- **Scalar Function:** A function that assigns a scalar value (a number) to each point in space. In Laplace's equation, u(x,y,z) is a scalar function.
- **Cartesian Coordinates:** A system for locating points in space using three perpendicular axes (x, y, and z), with the origin (0, 0, 0) at the intersection of these axes.
- **Laplacian Operator ∇^2**: An operator that represents the sum of the second partial derivatives of a function. In Cartesian coordinates, it is expressed as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

7.8 *REFERENCES:-*

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7.10 *TERMINAL QUESTIONS:-*

(TQ-1): Explain the steps involved in solving the Dirichlet problem for a rectangular domain. Provide an example scenario where this problem is encountered in real-world applications.

(TQ-2): Discuss the mathematical formulation of the Neumann problem for a rectangular domain. Provide insights into situations where Neumann boundary conditions are more appropriate than Dirichlet conditions.

(TQ-3): If $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ and $z = sinx$ at $y = 0$ for all values of x and $z = 0$ at $y = \infty$, for all values of x, show that $z = e^{-y} sinx$.

(TQ-4): If the edge of the breadth b of an infinitely long rectangular conducting strip is maintained at constant temperature T_0 the remaining edges being maintained at zero temperature. Show that the steady temperature distribution in the strip is given by

$$T = \frac{4T_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} sin\left\{(2n+1)\frac{\pi y}{b}\right\} e^{-(2n+1)\frac{\pi y}{b}}$$

(TQ-5): A thermally conducting solid bounded by two concentric spheres of radii a and b, (a < b), is such that the internal boundary is kept at temperature f1(θ) and the outer boundary at f2(θ). Find the steady state temperature in the solid.

7.11 *ANSWERS:-*

<u>SELF CHECK ANSWERS</u>

1. Dirichlet's Problem for a rectangle involves finding a harmonic function within the rectangle such that the function values are specified on the boundary of the rectangle.
2. The boundary conditions typically involve specifying the function values on all four sides of the rectangle.
3. Heat conduction in a rectangular region, where temperatures are specified on the boundaries, is an example.
4. Neumann's Problem for a rectangle involves finding a harmonic function within the rectangle such that the normal derivatives of the function are specified on the boundary.
5. The boundary conditions involve specifying the normal derivatives (partial derivatives with respect to the outward normal) on all four sides of the rectangle.

6. In fluid dynamics, Neumann's Problem for a rectangle can be applied to model the flow of a fluid through a rectangular channel with specified flux on the boundaries.
7. Yes, Dirichlet's Problem for a rectangle can be used to model the distribution of electric potential in a rectangular region with specified potentials on the boundaries.
8. Both problems are related to solving Laplace's equation, a second-order partial differential equation, in different boundary value contexts.
9. A harmonic function is a twice continuously differentiable function whose Laplacian is zero everywhere in its domain.

TERMINAL ANSWERS

(TQ-5): $\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B^n}{r^{n+1}}\right) P_n cos\theta$

Unit 8: Dirichlet's Problem and Newmann Problem for Circle

CONTENTS:

8.1 *INTRDUCTION:-*

Dirichlet's problem involves finding a harmonic function within a given domain that satisfies prescribed boundary values. Specifically, for a circle, the problem seeks a harmonic function inside the circle such that the function takes on given values on the circle's boundary. Neumann's problem focuses on determining a harmonic function within a domain when the normal derivative on the boundary is given. In the context of a circle, Neumann's problem aims to find a harmonic function inside the circle such that its normal derivative on the boundary (circle) matches a specified function.

These problems are fundamental in the study of partial differential equations and have broad applications in physics, engineering, and various scientific disciplines. The solutions to these problems provide insights into the behavior of harmonic functions and contribute to the understanding of potential theory.

8.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- Discuss the Dirichlet Problem for a Circle.
- Formulate the mathematical problem explicitly, defining the domain, the newmann problem for a circle.

By addressing these objectives, mathematicians aim to provide a comprehensive understanding of Dirichlet's Problem for a circle and contribute to the broader field of potential theory. The investigation of harmonic functions within circular domains has far-reaching implications in mathematical analysis and its applications. Similarly mathematicians aim to understanding of Neumann's Problem for a circle, contributing to the broader field of potential theory. The investigation of harmonic functions within circular domains, especially with prescribed conditions on the normal derivative, has implications in various mathematical and physical contexts.

8.3 *POLAR COORDINATES*$(\boldsymbol{r}, \boldsymbol{\theta})$*:-*

Now we given the Laplace equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \dots (1)$$

And we put $u(r,\theta) = R(r)\Theta(\theta)$ to given that

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k \quad \dots (2)$$

Where k is separation constant.

CaseI: Let $k = p^2 > 0, p$ being real, from (1), we have obtain

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - p^2R = 0, \frac{d^2\Theta}{d\theta^2} + p^2\Theta = 0$$

whose solutions are $R = c_1e^p + c_2e^{-p}$ and $\Theta = c_3 cosp\theta + c_4 sinp\theta$ respectively. Now from (1), we have

$$u(r,\theta) = (c_1e^p + c_2e^{-p})(c_3 cosp\theta + c_4 sinp\theta)$$

CaseII: Let $k = 0, p$ being real, from (2), we can be written as

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0, \frac{d^2\Theta}{d\theta^2} = 0$$

Obtaining the solutions as $R = c_5 Inr + c_6, \Theta = c_7 y + c_8$. Thus the equation (1) can be written as

$$u(r,\theta) = (c_5 Inr + c_6)(c_7 y + c_8) \qquad \dots (3)$$

Where c_j $(j = 5,6,7,8)$ are constants.

CaseIII: Let $k = -p^2 < 0,$ then the case I , the solution of the equation (1) is

$$u(r,\theta) = (c_9 \cos(pInr) + c_{10} sin(pInr))\left(c_{11} e^{p\theta} + c_{12} e^{-p\theta}\right) \quad \dots (4)$$

where c_j $(j = 1,2, \dots \dots 12)$ are constants.

8.4 *INTERIOR DIRICHLET PROBLEM FOR A CIRCLE:-*

The interior Dirichlet problem(Poisson's Formula) for a circle is described as below:

To find the value of single-valued and continuous function u within and on the circular region $r = a$ such that u satisfies the Laplace's equation (2) for $0 \le r \le a, 0 \le \theta \le 2\pi$ subject to the boundary condition $u(a,\theta) = f(\theta), 0 \le \theta \le 2\pi$, $f(\theta)$ being a continuous function of θ.

Since the function u is single-valued, it must satisfied the condition is

$$u = (r, \theta + 2\pi) = u(r,\theta), \qquad 0 \le \theta \le 2\pi \qquad \dots (5)$$

Let, $r = 0$, is a point of the domain of definition of the problem and Inr is undefined at $r = 0$. Thus the equation (3) and (4) of Laplace's equation are ruled out, therefore the equation (5) then obtain

$$c_3 \cos(p\theta) + c_4 sin(p\theta) = c_3 \cos(\theta + 2\pi) + c_4 sin(\theta + 2\pi)$$

$$c_3 \left[\cos(p\theta) - \cos(\theta + 2\pi)\right] + c_4 sin(p\theta) - c_4 sin(\theta + 2\pi) = 0$$

$$sinp\pi[c_3 sinp(\theta + \pi) - c_4 cosp(\theta + \pi)] = 0$$

Since $p = n, (n = 0,1,2 \dots)$

Hence, by using the superposition principle, the equation (1) is

$$u(r,\theta) = \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n cosn\theta + B_n sinn\theta)$$

Since $u(r,\theta)$ is be finite at the origin, so we put $D_n = 0$. Also taking $a_0 = 2A_0, a_n = 2A_nC_n, b_n = 2B_nC_n, (n > 0)$, the above solution can be written as

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sinh n\theta)$$

Now $u(a,\theta) = f(\theta)$ obtain

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a^n(a_n \cos n\theta + b_n \sinh n\theta)$$

$\Rightarrow$ $a_0 = \frac{1}{\pi}\int_0^{2\pi} f\,(\phi)d\phi, a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)cosn\phi d\phi,\ a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)sinn\phi\ d\phi,$

So , $u(r,\theta) = \frac{1}{\pi}\int_0^{\pi} f(\phi)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{n}\right)^n \cos(\phi-\theta)\right\}d\phi$

If we put $c = \sum_{n=1}^{\infty}\left(\frac{r}{n}\right)^n cosn(\phi-\theta)$ and $c = \sum_{n=1}^{\infty}\left(\frac{r}{n}\right)^n cosn(\phi-\theta)$

Then $c + is = \sum_{n=1}^{\infty}\left\{\left(\frac{p}{a}\right)e^{i(\phi-\theta)}\right\}^n = \frac{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}{1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}$

Equating real part

$$c = \sum_{n=1}^{\infty}\left(\frac{r}{n}\right)^n cosn(\phi-\theta) = \frac{\left(\frac{r}{a}\right)cos(\phi-\theta) - \frac{r^2}{a^2}}{1-\left(\frac{2r}{a}\right)cos(\phi-\theta)+\frac{r^2}{a^2}}$$

Hence the required solution is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)f(\phi)}{a^2-2arcos(\phi-\theta)+r^2}d\phi, \qquad r < a$$

This is called Poisson integral formula and obtains the interior Dirichlet problem for a circle.

8.5 *EXTERIOR DIRICHLET PROBLEM FOR A CIRCLE:-*

The exterior Dirichlet problem for a circle is described as below:

To find the value of single-valued and continuous function u within and on the circular region $r = a$ such that u satisfies the Laplace's equation (1) for $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$ subject to the boundary condition $u(a, \theta) = f(\theta), 0 \leq \theta \leq 2\pi, f(\theta)$ being a continuous function of θ and u is bounded $r \to \infty$.

Since the function u is bounded as $r \to \infty$, it must satisfied the condition is

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r^{-n} (a_n \cos n\theta + b_n \sinh n\theta) \qquad \dots (6)$$

Now $u(a, \theta) = f(\theta)$ obtain

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a^{-n} (a_n \cos n\theta + b_n \sinh n\theta)$$

$\Rightarrow \qquad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\phi) d\phi, a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\phi) cosn\phi d\phi, \; a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\phi) sinn\phi \, d\phi$

So , $\quad u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{a}{r} \right)^n cosn(\phi - \theta) \right\} d\phi$

Then the case of interior Dirichlet problem, we get

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - a^2) f(\phi)}{r^2 - 2arcos(\phi - \theta) + a^2} d\phi, \qquad r > a$$

This is called the exterior Dirichlet problem for a circle.

8.6 *INTERIOR NEWMANN PROBLEM FOR A CIRCLE:-*

The Exterior Interior Neumann for a Circle is described as below:

To obtain the value of single-valued and continuous function u within and on the circular region $r = a$ such that u satisfies the Laplace's equation (1) for $0 \le r \le a, 0 \le \theta \le 2\pi$ subject to the boundary condition $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = (\theta)$, on $r = a$ where $g(\theta), 0 \le \theta \le 2\pi$ and u is bounded θ.

Since the function u is bounded as $r = 0$, it must satisfied the condition is

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sinh n\theta) \qquad \dots (1)$$

Now

$$u(r,\theta) = \frac{1}{2}a + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sinh n\theta)$$

Where $A_0 = \frac{1}{2}a, A_n = a_n, B_n = b_n, (n > 0)$. So we have

$$g(\theta) = \sum_{n=1}^{\infty} na^{n-1}(a_n \cos n\theta + b_n \sin n\theta)$$

Which is full-range Fourier series in $g(\theta)$. Thus

$$a_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi) cosn\phi d\phi,\ \ b_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi) sinn\phi d\phi$$

And the equation (1) obtain as

$$u(r,\theta) = \frac{1}{2}a + \int_0^{2\pi} g\,(\phi)\left\{\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{a}{n\pi} cosn(\phi-\theta)\right\} d\phi$$

Substituting

$c = \sum_{n=1}^{\infty} \frac{a}{n\pi}\left(\frac{r}{n}\right)^n cosn(\phi-\theta)$ and $c = \sum_{n=1}^{\infty} \frac{a}{n\pi}\left(\frac{r}{n}\right)^n cosn(\phi-\theta)$

Then $\quad c + is = \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right\}^n = -\frac{a}{\pi} In\left\{1 - \frac{r}{a}e^{i(\phi-\theta)}\right\}$

Equating real part

$$c = -\frac{a}{2\pi} In\{a^2 - 2arcos(\phi-\theta) + r^2/a^2\}$$

Hence the required solution is

$$u(r,\theta) = \frac{1}{2a} - \frac{a}{2\pi}\int_0^{2\pi} In\left\{a^2 - \frac{2r}{a}cos(\phi-\theta) + \frac{r^2}{a^2}\right\} g(\phi)\, d\phi,$$
$$r < a$$

8.7 *SPHERICAL POLAR COORDINATES:-*

In the case of axial symmetry about the polar axis ($\theta = 0$), a function $\psi(r,\theta,\phi)$ is said to be independent of ϕ and Laplace's equation in spherical polar coordinates is given below

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\psi}{\partial r}\right) = 0 \qquad ..(1)$$

Now we put $\psi(r,\theta) = R(r)\Theta(\theta)$ ... (2)

Where R and Θ are the functions of r and θ and the function $\Theta(\theta)$ is called the zonal surface harmonic. From (1) and (2), we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) = k$$

Where k is separation constant. Putting $k = n(n+1)$, we obtain

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \quad ... (3)$$

and

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0 \quad ... (4)$$

From (3), we get

$$R(r) = Ar^r + \frac{B}{r^{r+1}} \qquad ... (5)$$

Now again putting $\mu = cos\theta$,from (4) we obtain

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$$

This is called Lagrange's equation whose solution is

$$\Theta(\theta) = CP_n(\mu) + DQ_n(\mu) \qquad ... (6)$$

where the functions $P_n(\mu)$ and $Q_n(\mu)$ are Lagrange's functions of the first and second kind respectively.

Using (5) and (6) in equation(1) and (2) ,we get

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\{CP_n(\mu) + DQ_n(\mu)\}$$

SELF CHECK QUESTIONS

1. What is Dirichlet's Problem for a circle?
2. State the mathematical formulation of Dirichlet's Problem for a circle.
3. What is the Neumann Problem for a circle?
4. What is the relationship between Dirichlet's and Neumann Problems for a circle?
5. Can Dirichlet's Problem have a unique solution for any given boundary condition?
6. What are the boundary conditions for Dirichlet's Problem and Neumann Problem for a circle?

8.8 *SUMMARY:-*

In this unit we have studied the Dirichlet's Problem and Newmann Problem for Circle. Both problems involve finding solutions to partial differential equations within a circular domain, but the Neumann problem focuses on prescribing the behavior of the normal derivative on the boundary, while the Dirichlet problem focuses on prescribing the actual values of the solution on the boundary.

8.9 *GLOSSARY:-*

Neumann Problem for a Circle:

1. **Mathematical Setup:**
    - **Circular Domain:** A region in the two-dimensional plane defined by a circular boundary.
    - **Laplace Equation:** A partial differential equation describing the distribution of a scalar field within the circular domain. It is often denoted as $\nabla^2 u = 0$ and $\nabla^2 u = f(x, y)$, u is unknown function.

2. **Neumann Boundary Conditions:**
   - **Normal Derivative:** The rate of change of the solution along the normal direction to the circular boundary.
   - **Neumann Conditions:** Prescribed normal derivatives on the circular boundary, typically represented as $\frac{\partial u}{\partial n} = g(\theta)$, where θ is the angle parameter.
3. **Solution Procedure:**
   - **Separation of Variables:** A mathematical technique used to simplify the solution process by expressing the solution as a product of functions of individual variables.
   - **Conformal Mapping:** A transformation method that preserves angles and shapes, often applied to simplify the solution process for circular geometries.
4. **Circular Boundary:** The outer boundary of the circular domain within which the Neumann problem is defined.
5. **Angle Parameter ($\boldsymbol{\theta}$):** A parameterization of points along the circular boundary, often measured in radians.
6. **Scalar Field ($\boldsymbol{u}$):** The unknown function representing the physical quantity being studied (e.g., temperature, electric potential) within the circular domain.

Dirichlet Problem for a Circle:

1. **Dirichlet Boundary Conditions:**
   - **Dirichlet Conditions:** Prescribed values of the solution on the circular boundary, typically represented as $u(\theta) = h(\theta)$, where $u(\theta)$ is the solution at the point parameterized by the angle θ, and $h(\theta)$ is a given function specifying the prescribed values.
2. **Solution Procedure:**
   - **Separation of Variables:** A mathematical technique used to simplify the solution process by expressing the solution as a product of functions of individual variables.
   - **Conformal Mapping:** A transformation method that preserves angles and shapes, often applied to simplify the solution process for circular geometries.
3. **Circular Boundary:** The outer boundary of the circular domain within which the Dirichlet problem is defined.
4. **Angle Parameter ($\boldsymbol{\theta}$):** A parameterization of points along the circular boundary, often measured in radians.

5. **Scalar Field ($\boldsymbol{u}$):** The unknown function representing the physical quantity being studied (e.g., temperature, electric potential) within the circular domain.

In summary, both Neumann and Dirichlet problems involve finding solutions to the Laplace or Poisson equation within a circular domain, with Neumann conditions specifying normal derivatives and Dirichlet conditions specifying actual values on the circular boundary. Techniques such as separation of variables and conformal mapping are commonly used in solving these problems.

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- https://sites.millersville.edu/rbuchanan/math467/LaplaceNeumann.pdf
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- David L. Powers(2010), Boundary Value Problems and Partial Differential Equations.

8.12 *TERMINAL QUESTIONS:-*

(TQ-1): Show that the Laplace's equation $\nabla^2 u = 0$ in cylindrical coordinates satisfying the conditions (i) $u \to 0\ as\ z \to 0$ (ii) u is infinite as $\rho \to 0$ are of the form.

$$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm in\phi}$$

(TQ-2): Show that the two dimensional Laplace equation $\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}=0$ has solutions of the form $(Ar^n+Br^{-n})e^{\pm in\theta}$, where A and B and n are constants. Determine V satisfying the above equation in the region $0\le r\le a, 0\le\theta\le 2\pi$ and satisfying the boundary conditions.

i. V remains finite as $r\to 0$.
ii. $V=\sum_n C_n \cos n\theta$ on $r=a$.

(TQ-3): Discuss the Interior Neumann Problem for a circle formulation Mathematically

(TQ-4): Explain the concept of spherical polar coordinates in detail, particularly focusing on their application in Laplace's equation.

(TQ-5): Explore the concepts of the exterior and interior Dirichlet problem, focusing specifically on their application to a circular domain.

8.13 *ANSWERS:-*

SELF CHECK ANSWERS

1. Dirichlet's Problem involves finding a harmonic function inside a region that takes specified values on the boundary of that region. For a circle, it is the task of finding a harmonic function within the circle that matches prescribed values on its boundary.
2. Let D be the unit disc in the complex plane, and f(θ) be a given continuous function defined on the boundary of D(∂D). Dirichlet's Problem is to find a harmonic function u(z) in D such that$u(e^{i\theta})=f(\theta)\ \forall\ \theta\in[0,2\pi)$.
3. The Neumann Problem involves finding a harmonic conjugate of a given function on the boundary of a region. For a circle, it is the task of finding a harmonic conjugate for a given function defined on the circle's boundary.
4. For a simply connected domain (like a circle), the solutions to Dirichlet's Problem and Neumann Problem are related through the Cauchy-Riemann equations. If $u(z)$ is a solution to Dirichlet's Problem, then its harmonic conjugate $v(z)$ is a solution to the Neumann Problem, and vice versa.
5. No, Dirichlet's Problem may not have a unique solution in general. Uniqueness depends on the regularity of the boundary condition specified on the boundary of the region.

6. For Dirichlet's Problem, the boundary condition is a prescribed function $f(\theta)$ on the boundary of the circle. For Neumann Problem, the boundary condition is the derivative of a given function $g(\theta)$ on the circle's boundary.

Unit 9: Diffusion Equation

CONTENTS:

9.1 *INTRODUCTION:-*

The diffusion equation as a partial differential equation describing the collective motion of micro-particles resulting from their random movement within a material. The mention of its applications in mathematics, particularly in connection with the Markov process, and its wide-ranging relevance in fields like materials sciences, information science, life science, and social science highlights the interdisciplinary nature of the diffusion equation. The term "Brown problems" is introduced, connecting the diffusion equation to Brownian motion and indicating a class of problems related to the behavior of particles undergoing random motion. The discussion on the continuity of the diffusion equation in both space and time emphasizes the need for discretization in practical applications. The different approaches to discretization time alone, space alone, and both time and space are briefly outlined, with a mention of how discretization may lead to phenomena such as the random walk.

This unit, we will study a focus on the study of diffusion equations and their elementary solutions, suggesting a deeper exploration into the mathematical aspects and fundamental solutions of the diffusion equation. This introduction effectively sets the stage for a comprehensive understanding of this important concept in mathematical modeling and scientific research.

9.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- Solution of Diffusion equation.
- Explain Diffusion Equation and its Elementary Solution.

These objectives suggest that learners will gain both theoretical and practical knowledge in solving diffusion equations and understanding their elementary solutions. The skills acquired can be applied to various fields where diffusion phenomena are relevant, such as physics, chemistry, engineering, biology, and environmental science.

9.3 *ONE DIMENSIONAL (HEAT EQUATION) DIFFUSION EQUATION:-*

Let us consider that the temperature at any point (x, y, z) of a solid at time t is u (x, y, z, t) and the thermal conductivity of the solid is K, the density of the solid is p and specific heat is σ, the heat equation

$$\frac{\partial^2 u}{\partial t^2} = h^2 \nabla^2 u \qquad \dots (1)$$

Where $h^2 = \frac{K}{\rho\sigma} = k, k$ is known as diffusivity, is called the equation of diffusion or the Fourier equation of heat flow.

Suppose the flow of heat by conduction in a bar OA. Let be taken as the $x - axis$. Now an element $PQQ'P'$ of the bar as shown in figure. Let us suppose the temperature $u(x, t)$ of the bar at any point P is the function of x and the time t. Suppose that the bar is increased to an temperature distribution at the time $t = 0$ and then heat is allowed to flow of conduction.

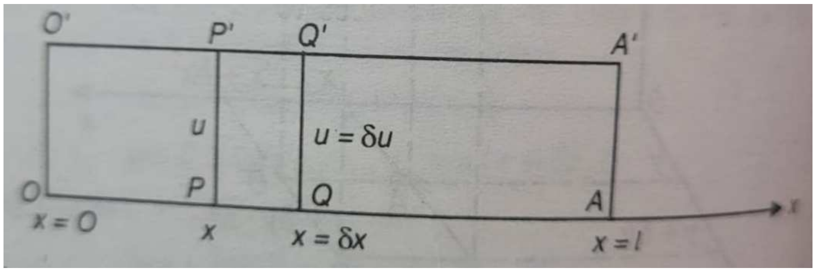

Fig.1

Let we constitute the following assumption.

1. The bar is homogeneous, i.e.,the mass of the bar unit volume is constant ρ.
2. The sides of the bar are offended and the loss of heat from the sides by conduction or radian can be neglected.
3. The amount of heat crossing any section of bar is obtained by $\frac{kA\partial u}{\partial x}\delta t$, where

$A =$ area of cross section of bar

$\frac{\partial u}{\partial x} =$ temperature gradient at the section

$\delta t =$ time of flow of heat

$K =$ thermal conductivity

Now the element across the section PP' in term δt

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \delta t$$

where the negative sign show the heat flow in the direction o decreasing temperature

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

Hence, the quantity of heat is

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \delta t - KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

$$= KA\delta t\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} \qquad \dots(2)$$

Let the above equation the temperature of the element by a small quantity δu. Then the same quantity of heat is again obtained by

$$= (\rho A\delta x)\sigma\delta u \qquad \dots (3)$$

Where σ is specific heat of bar.

From (2) and (3), we get

$$KA\delta t\{u(x + \delta x, t) - u(x, t)\} = (\rho A\delta x)\sigma\delta u$$

Or

$$K\frac{u(x + \delta x, t) - u(x, t)}{\delta x} = \rho\sigma\frac{\delta u}{\delta t} \qquad \dots (4)$$

Now as $\delta x \to 0$ and $\delta t \to 0$, reduces to

$$K\frac{\partial^2 u}{\partial x^2} = \rho\sigma\frac{\delta u}{\delta t} \qquad \text{or} \qquad k\frac{\partial^2 u}{\partial x^2} = \frac{\delta u}{\delta t} \qquad \dots (5)$$

where $\frac{K}{\rho\sigma} = k$ is called the diffusivity of the material of bar and equation (5) is known as one dimensional diffusion equation.

9.4 *DERIVATION OF FOURIER EQUATION OF HEAT CONDUCTION:-*

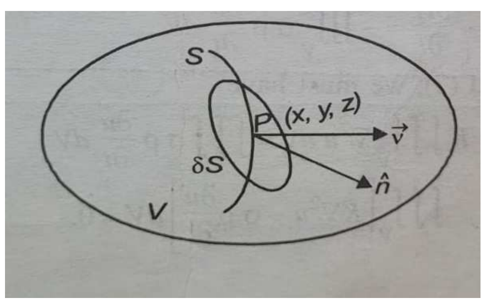

Fig.2

From the figure, let $P(x, y, z)$ be any point of the solid and δS be elementary area of S surrounding P. Let $\hat{n}$ be the unit outward drawn vector, normal at P to δS.

Now, the amount of heat leaving δS per unit time

$$= (\vec{v}.\vec{n})\ \delta S \qquad \dots (1)$$

$\therefore \qquad v = -K\, gard\, u = -K\nabla u \qquad \text{... (2)}$

where K is thermal conductivity of the material of the solid body and $u(x, y, z, t)$ is temperature at $P(x, y, z)$ at the time t.Hence

$$\nabla u = gard\, u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z} \qquad \text{... (3)}$$

From (1) and (2), we get

$$= -K \iint_S \nabla u.\widehat{n}\ dS \qquad \text{... (4)}$$

Now we consider the gauss divergence theorem, then we can be written as below

$$\iint_S F.\hat{n}\, dS = \iiint_V \nabla.F dV. \qquad \text{... (5)}$$

Let the total amount of heat leaving the entire surface S per second from (4) is obtained as

$$-K \iiint_V \nabla.\nabla u\ dV = -K \iiint_V \nabla^2 u\ dV \qquad \text{... (6)}$$

Let σ be the specific heat of the material of the body, then the total amount of heat H inside the surface S is obtained by

$$H = \iiint_V \sigma\rho u\ dV \qquad \text{... (7)}$$

Hence the rate of decrease of H inside the entire surface S $i.e.$, the amount of heat leaving the entire volume V across its surface S per second is

$$-\frac{\partial H}{\partial t} = -\iiint_V \sigma\rho u\frac{\partial u}{\partial t}\ dV \qquad \text{... (8)}$$

From (6) and (8), we get

$$K \iiint_V \nabla^2 u\ dV = \iiint_V \sigma\rho\frac{\partial u}{\partial t}\ dV$$

$$\iiint_V \nabla^2 u\ dV \left[K\nabla^2 u - \sigma\rho\frac{\partial u}{\partial t}\right] dV = 0 \qquad \text{... (9)}$$

V being arbitrary and integrated in (9), if its integrated is zero everywhere, i.e.,

$$K\nabla^2 u - \sigma\rho\frac{\partial u}{\partial t} = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{K}{\sigma\rho}\nabla^2 u$$

Or

$$\frac{\partial u}{\partial t} = K\nabla^2 u$$

where $k = \frac{K}{\sigma\rho}$ is thermal diffusivity of the material.

9.5 *TELEGRAPGH OR TRANSMISSON LINE EQUATIONS:-*

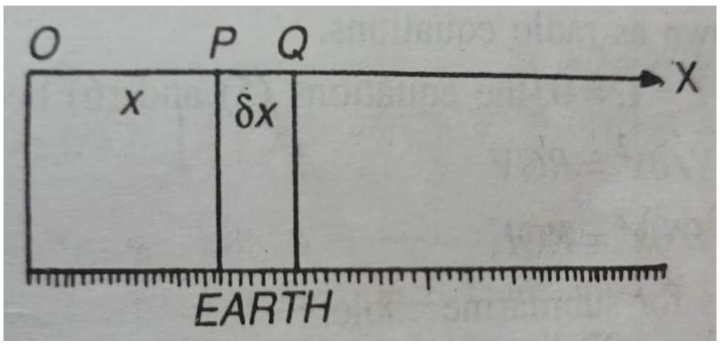

Fig.3

As shown in figure 3 considering the fall of potential in a linear element $PQ(-\delta x)$ situated at point x, we obtain

$$-\delta V = IR\delta x + L\delta x\left(\frac{\partial V}{\partial t}\right)$$

Where R and L are resistance and inductance per unit length respectively.

If there is a capacitance to the earth of C per unit length and conductance G per unit length , then

$$-\delta I = GV\delta x + C\delta x(\partial V/\partial t)$$

Now $\delta x \to 0$, from the above equations,

$$\frac{\partial V}{\partial x} + RI + L\frac{\partial I}{\partial t} = 0 \qquad \dots (1)$$

and

$$\frac{\partial I}{\partial x} + GV + \frac{C\partial V}{\partial t} = 0 \quad \dots (2)$$

Differentiating (1), w.r.t.x and (2) w.r.t. t , we obtain

$$\frac{\partial^2 V}{\partial x^2} + r\frac{\partial I}{\partial x} + L\frac{\partial^2 I}{\partial x \partial t} = 0$$

$$G\frac{\partial V}{\partial t} + C\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 I}{\partial x \partial t} = 0$$

Eliminating $\frac{\partial I}{\partial x}$ and $\frac{\partial^2 I}{\partial x \partial t}$ from the above equations , we get

$$\frac{\partial^2 V}{\partial x^2} = CL\left(\frac{\partial^2 V}{\partial t^2}\right) + (CR + GL)\frac{\partial V}{\partial t} + RGV \qquad \dots (3)$$

Similarly differentiating (1) w.r.t. t and (2) w.r.t. x, we have

$$\frac{\partial^2 I}{\partial x^2} = CL\left(\frac{\partial^2 I}{\partial t^2}\right) + (CR + GL)\frac{\partial I}{\partial t} + RGI \quad \dots (4)$$

Hence this equation is known as telephone equations.

Remark1: If $L = G = 0$, the equation (3) and (4), we obtain

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}$$

$$\frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}$$

is called **telephone equations.**

Remark2: If $R = G = 0$, the equation (3) and (4), we get

$$\frac{\partial^2 V}{\partial x^2} = CL\left(\frac{\partial^2 V}{\partial t^2}\right)$$

$$\frac{\partial^2 I}{\partial x^2} = CL\left(\frac{\partial^2 I}{\partial t^2}\right)$$

This is called **radio equations.**

Remark3: If $L = C = 0$, the equation (3) and (4), we have

$$\frac{\partial^2 V}{\partial x^2} = RGV$$

$$\frac{\partial^2 I}{\partial x^2} = RGI$$

This equation called **submarine cable.**

9.6 *BOUNDARY VALUE PROBLEM:-*

A boundary value problem (BVP) is a type of mathematical problem that involves finding a solution to a differential equation subject to certain conditions on the boundary of the domain in which the equation is defined. These problems are common in various fields, including physics, engineering, and applied mathematics. Boundary value problems are distinct from initial value problems, which involve specifying conditions at a single point within the domain.

Mathematically, a Boundary Value Problem can be expressed as follows:

Consider a differential equation of the form:

$$F\ x, y, y'y'' \dots., y^n = 0$$

subject to boundary conditions:

$$G\ y(a), y(b), y'(a), y'(b), \dots \dots., y^{(n-1)}(a) y^{(n-1)}(b) = 0$$

Here y in unknown function, y' represents the first derivative of y, y'' the second derivative, and so on, up to the n^{th} derivative. The function F represents the differential equation, and G represent the boundary condition.

The solution to a Boundary Value Problem seeks to find a function $y(x)$ that satisfies both the differential equation and the specified boundary conditions within the obtained domain $[a, b]$.

9.7 *METHOD OF SEPERATION OF VARIABLES OR PRODUCT METHOD:-*

The method you're describing is known as the method of separation of variables, commonly used in solving partial differential equations (PDEs).

Suppose a partial differential equation (PDE) involving n independent variables n independent variable $x_1, x_2, \ldots\ldots x_n$ and one dependent variable u.Then we first assume that the solution to this PDE can be represented as a product of functions of the form

$$u(x_1, x_2, \ldots\ldots x_n) = X_1(x_1), X_2(x_2), \ldots\ldots X_n(x_n) \qquad \ldots (1)$$

where X_i is a function of x_i only $(i = 1,2, \ldots\ldots n)$. On putting of (1) into the given equation, we shall obtain n ordinary differential equations one in each of unknown functions $X_i (= 1,2, \ldots., n)$. The entire procedure will be clear from the solved examples.

9.8 *THE PRINCIPLE OF SUPERPOSITION:-*

The general linear homogeneous partial differential equation of the second order is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad \ldots (1)$$

Let us suppose that

i. $u_1, u_2, \ldots\ldots u_n, \ldots.$ is an infinite set of solutions of (1) in region R in $xy -$ plane .
ii. The infinite series $u_1 + u_2 + \cdots + u_n + \cdots$ converges and is differentiable term by term in R. Then by the principle of superposition, the function u, defined by $u = \sum_{n=1}^{\infty} u_{n,}$ is also a solution of (1) in R.

9.9 *INITIAL AND BOUNDARY CONDITIONS:-*

The conditions which are obtained for time $t = 0$ are called initial conditions (I.C). The conditions obtained at the boundary of the region or intervals are called boundary condition (B.C).

Let us suppose $u(x, t)$ and $v(x, y, t)$ are functions of two and three variable respectively. Then, we have

$$\frac{\partial u}{\partial x} = u_x = u_x(x, t), \qquad \frac{\partial u}{\partial t} = u_t = u_t(x, t)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = u_x(\pi, t), \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = u_t(x, 0)$$

$$\frac{\partial v}{\partial x} = v_x(x, y, t), \qquad \left(\frac{\partial v}{\partial t}\right)_{t=0} = v_t(x, y, 0)\ etc.$$

EXAMPLE: Obtain the general solution of heat flow equation $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ by the method of separations of variables

SOLUTION: Given $\qquad k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad$ ... (1)

Now from (1), we get

$$u(x, t) = X(x)T(t) \qquad \text{... (2)}$$

Now then find, on putting in (1), we get

$$kX''T = XT' \text{ or } \frac{X''}{X} = \frac{T'}{kT} \qquad \text{... (3)}$$

where the dashes denote derivatives with respect to the relevant variable. From (3), we have

$$X'' - \mu X = 0 \qquad \text{... (4)}$$

and

$$T' = \mu kT \qquad \text{... (5)}$$

Since the three cases according as μ is zero, $+ve$ or $-ve$.

CaseI: Suppose $\mu = 0$, then the solution of (4) and (5) are

$$X = a_1x + a_2 \text{ and } \quad T = a_3$$

CaseII: Let μ be $+ve$, say λ^2. Then from (4) and (5) becomes respectively $X'' - \lambda^2 X = 0$ and $T' = \lambda^2 kT$, we obtain

$$X = b_1e^{\lambda x} + b_2e^{-\lambda x} \quad and \quad T = b_3e^{\lambda^2 kT}$$

CaseIII: Let μ be $-ve$, say $-\lambda^2$. Then from (4) and (5) becomes respectively $X'' + \lambda^2 X = 0$ and $T' = -\lambda^2 kT$, we get

$$X = c_1cos\lambda x + c_2sin\lambda x \quad and \quad T = c_3e^{-\lambda^2 kT}$$

Hence , $\qquad u(x, t) = A_1x + A_2$

$$u(x,t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 kt}$$

$$u(x,t) = (C_1 cos\lambda x + C_2 sin\lambda x) e^{-\lambda^2 kt}$$

where $A_1 = a_1 a_3, A_2 = a_2 a_3, B_1 = b_1 b_3, B_2 = b_2 b_3, C_1 = c_1 c_3$ and $C_2 = c_2 c_3$ are new arbitrary constants.

9.10 *TWO DIMENSIONAL (HEAT EQUATION) DIFFUSION EQUATION:-*

The two-dimensional heat equation describes how temperature distribution evolves over time in a two-dimensional space. It is a partial differential equation that represents the diffusion of heat in two spatial dimensions. The general form of the two-dimensional heat equation is given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \dots (1)$$

- $u(x,y,t)$ is the temperature distribution at the point(x,y) and t.
- $\frac{\partial u}{\partial t}$ is the partial derivative of u with respect to time, representing the rate of change of temperature.
- $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ are the second partial derivatives of u w.r.t. x and y respectively, representing the spatial variations of temperature in each dimension.
- $\frac{1}{k}$is the thermal diffusivity, a material property that characterizes how quickly heat can spread through the medium.

Problem: Obtain the solution of the two- dimensional diffusion equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{k}\frac{\partial \theta}{\partial t} \qquad \dots (1)$$

Sol: Let that the equation (1) has solutions of the form

$$\theta(x,y,t) = X(x)Y(y)T(t) \qquad \dots (2)$$

Putting this value of θ in (1), we obtain

$$X''YT + XY''T = \frac{1}{k} XYT' \ \ or \quad \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{k}\frac{T'}{T} \qquad \dots (3)$$

Since x, y and t are independent variables, (3) is true if

$$\frac{X''}{X} = -n^2, \frac{Y''}{Y} = -m^2 \ \ and \ \ \frac{T\prime}{kT} = -p^2 \qquad \dots(4)$$

with

$$n^2 + m^2 = p^2$$

Now solving differential equation obtained by (4) $(\theta \to 0 \ as \ t \to \infty)$

$$X_n(x) = A_n cosnx + B_n sinnx,\ Y_m(y) = C_n \cos\ my + D_m sin\ my,$$

And

$$T_p(t) = E_p e^{-p^2 kt} = F_{nm} e^{-(n^2+m^2)kt}$$

Hence

$\therefore \qquad \theta_{nm}(x, y, t) = F_{nm}(A_n cosnx + B_n sinnx) \times (C_n \cos\ my + D_m sin\ my) e^{-(n^2+m^2)kt}$.

9.11 *THREE DIMENSIONAL (HEAT EQUATION) DIFFUSION EQUATION:-*

The three-dimensional heat equation describes how temperature distribution evolves over time in a three-dimensional space. It is a partial differential equation that represents the diffusion of heat in three spatial dimensions. The general form of the three-dimensional heat equation is given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \dots(1)$$

Suppose the equation (1) has solution of the form is given by

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t) \qquad \dots(2)$$

where X, Y, Z and T are respectively the functions of x, y, z and t alone. Putting the value of u in (1), we obtain

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{k}\ XYZT'$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k}\frac{T'}{T} \qquad \dots(3)$$

Since x, y, z, t are independent variables, so

$$\frac{X''}{X} = -n^2, \frac{Y''}{Y} = -m^2, \frac{Z''}{Z} = -l^2, \ \frac{T'}{kT} = \ -p^2 \quad \ldots (4)$$

where

$$n^2 + m^2 + l^2 = p^2 \quad \ldots (5)$$

Again from (4), we obtain

$$X_n(x) = A_n cosnx + B_n sinnx, \qquad Y_n(y) = C_n \cos\, my + D_m sin\, my$$

$$Z_1(z) = E_l \cos lz + F_l sinlz$$

and

$$T_p(t) = G_p e^{-p^2kt} = H_{nml} e^{-(\,n^2+m^2+l^2)kt}$$

$\therefore \quad u_{nml}(x, y, z, t) = H_{nml}(A_n cosnx + B_n sinnx)(C_n \cos\, my + D_m sin\, my)(E_l \cos lz + F_l sinlz)e^{-(\,n^2+m^2+l^2)kt} \quad \ldots (6)$

are the solutions of (1). Hence the equation (10 is obtained by

$$u(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} u_{nml}(x, y, z, t).$$

<u>SELF CHECK QUESTIONS</u>

1. What is the diffusion equation?
2. What does the diffusion coefficient (D) represent in the diffusion equation?
3. What is the one-dimensional diffusion equation?
4. What is the two-dimensional diffusion equation?
5. What is the three-dimensional diffusion equation?
6. What are boundary value problems (BVPs) in the context of the diffusion equation?
7. What is the general form of the diffusion equation, and what does it describe?
8. In the context of the diffusion equation, what does the term ∂u/∂t represent?

9.12 *SUMMARY:-*

The unit covers various aspects of the diffusion equation, exploring its applications in different dimensions. It begins by examining the general diffusion equation and then delves into specific cases such as the one-dimensional, two-dimensional, and three-dimensional diffusion equations.

The unit also discusses boundary value problems (BVP), providing a comprehensive understanding of the mathematical models and solutions associated with diffusion processes in different scenarios. Overall, the unit encompasses a thorough study of diffusion phenomena, offering insights into their mathematical formulations and solutions across various dimensions and boundary conditions.

9.13 *GLOSSARY:-*

- **Diffusion Equation:** A partial differential equation that describes the distribution of a quantity (e.g., temperature, concentration) over time and space as it spreads through a medium.
- **One-Dimensional Diffusion Equation:** A specific form of the diffusion equation applied to situations where the spreading of a substance occurs in only one spatial dimension.
- **Two-Dimensional Diffusion Equation:** A variant of the diffusion equation used to model the diffusion process in two spatial dimensions.
- **Three-Dimensional Diffusion Equation:** The diffusion equation adapted for situations where the spreading of a substance occurs in three spatial dimensions.
- **Boundary Value Problems (BVP):** A type of mathematical problem associated with finding a solution to a differential equation that satisfies certain conditions at the boundaries of the domain.
- **Concentration Profile:** The spatial distribution of the concentration of a substance over a given area or volume as it undergoes diffusion.
- **Diffusivity (D):** A parameter that represents the rate at which a substance diffuses through a medium. It is a measure of how easily a substance spreads.
- **Initial Condition:** The concentration distribution at the starting point in both space and time for a diffusion process.
- **Dirichlet Boundary Condition:** A type of boundary condition where the concentration of the diffusing substance is specified at certain points on the boundary of the domain.
- **Neumann Boundary Condition:** A type of boundary condition where the flux (rate of flow) of the diffusing substance is specified at certain points on the boundary.

- **Steady-State Solution:** A solution to the diffusion equation where the concentration profile does not change with time, indicating a constant state of diffusion.
- **Transient Solution:** A solution to the diffusion equation that considers changes in concentration over time, capturing the dynamic nature of the diffusion process.
- **Finite Difference Method:** A numerical technique commonly used to solve differential equations, including the diffusion equation, by approximating derivatives with finite differences.
- **Analytical Solution:** A solution to the diffusion equation obtained through mathematical analysis and manipulation of the equation, providing an exact expression for the concentration profile.
- **Numerical Simulation:** The use of computational methods to approximate solutions to the diffusion equation when analytical solutions are difficult or impossible to obtain.

These terms collectively contribute to the understanding and analysis of diffusion phenomena in different dimensions and under various conditions.

9.14 *REFERENCES:-*

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9.15 *SUGGESTED READING:-*

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9.16 *TERMINAL QUESTIONS:-*

(TQ-1): Solve B.V.P. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, if $u(0, y) = 8e^{-3y}$.

(TQ-2): Solve by the method of separation of variables

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

(TQ-3): If both the ends of a bar of length a are at temperature zero and the initial temperature is two be prescribed function $f(x)$ in the bar, then find the temperature at subsequent time t.

(TQ-4): A rod of length l with insulted sides, is initially at a uniform temp u_0 . Its are suddenly cooled to 0^0C and are kept at that temperature. Find the temperation $u(x, t)$.

(TQ-5): Find the meth of separation of variables the solution $u(x, t)$ of the boundary value problem.

$$\frac{\partial U}{\partial t} = 3\left(\frac{\partial^2 U}{\partial x^2}\right), t > 0, 0 < x < 2,$$

$$U(0, t) = 0, U(2, t) = 0, t > 0; U(x, 0) = x, 0 < x < 2.$$

(TQ-6): A rectangular plate bounded by the lines $x = 0, y = 0, x$ $a, y = b$ has an initial distribution given by $V = Asin(\pi x/a)sin(\pi x/b)$. The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature at any point.

9.17 *ANSWERS:-*

<u>SELF CHECK ANSWERS</u>

1. The diffusion equation describes the spread of substances through a medium and is mathematically expressed as $\partial C/\partial t = D\nabla^2 C$, where C is concentration, t is time, D is the diffusion coefficient, and ∇^2 is the Laplacian operator.
2. The diffusion coefficient (D) represents the rate at which a substance diffuses through a medium, indicating how quickly molecules spread and mix.
3. The one-dimensional diffusion equation is $\partial C/\partial t = D\partial^2 C/\partial x^2$, where C is concentration, t is time, D is the diffusion coefficient, and $\partial^2 C/\partial x^2$ is the second partial derivative of concentration with respect to spatial coordinate x.

4. The two-dimensional diffusion equation is $\partial C/\partial t = D(\partial^2 C/\partial x^2 + \partial^2 C/\partial y^2)$, where C is concentration, t is time, D is the diffusion coefficient, and $\partial^2 C/\partial x^2$ and $\partial^2 C/\partial y^2$ are the second partial derivatives of concentration with respect to spatial coordinates x and y, respectively.
5. The three-dimensional diffusion equation is $\partial C/\partial t = D(\partial^2 C/\partial x^2 + \partial^2 C/\partial y^2 + \partial^2 C/\partial z^2)$, where C is concentration, t is time, D is the diffusion coefficient, and $\partial^2 C/\partial x^2, \partial^2 C/\partial y^2$, and $\partial^2 C/\partial z^2$ are the second partial derivatives of concentration with respect to spatial coordinates x, y, and z, respectively.
6. Boundary value problems (BVPs) involve specifying conditions on the boundaries of the spatial domain in addition to the initial conditions. These conditions could be Dirichlet conditions (specifying the value of u at certain boundaries) or Neumann conditions (specifying the derivative of u at certain boundaries).
7. The general form of the diffusion equation is $\partial u/\partial t = D\nabla^2 u$, where u is the scalar field representing the quantity undergoing diffusion, t is time, and D is the diffusion coefficient. It describes how the quantity u spreads over time due to concentration differences.
8. $\partial u/\partial t$ represents the rate of change of the quantity u with respect to time. It quantifies how the concentration of the diffusing substance changes over time.

TERMINAL ANSWERS

(TQ-1): $u(0,y) = 8e^{-(4x+y)}$.

(TQ-2): $z(x,y) = \left\{De^{[1+\sqrt{1+k}]x} + Ee^{[1+\sqrt{1-k}]x}\right\}e^{ky}$.

(TQ-4): $u(x,t) = \frac{4u_0}{\pi}\sum_{m=1}^{\infty} E_{2m-1}\, sin\frac{(2m-1)}{l}\pi x\; e^{-C_{2m-1}^2 t}$

(TQ-5): $U(x,t) = \sum_{n=1}^{\infty}\left(\frac{\sin\frac{n\pi x}{2}}{\sin\frac{n\pi}{2}}\right)e^{-3n^2\pi^2 t/4}$

(TQ-6): $u(x,y,t) = Asin\left(\frac{\pi x}{a}\right)sin\left(\frac{\pi y}{a}\right)e^{-\pi^2 kt\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$

BLOCKIII
PARTIAL DIFFERENTIAL EQUATIONS-III

Unit 10: Wave Equation

CONTENTS:

10.1 *INTRODUCTION:-*

A wave is initiated when a vibrating source periodically disturbs the first particle of a medium. This disturbance creates a wave pattern that travels through the medium from particle to particle. The frequency of vibration for each individual particle matches the frequency of the source, and the period of vibration for each particle is equal to the source's period. In one complete period, the source displaces the first particle upwards, returns it to rest, moves it downwards, and brings it back to rest, completing one full back-and-forth cycle, which constitutes a single wave cycle

The wave equation is a crucial second-order linear partial differential equation used to describe various types of waves in classical physics, such as mechanical waves (e.g., water waves, sound waves, seismic waves) and light waves. It finds applications in fields like acoustics, electromagnetic, and fluid dynamics. The historical development of the wave equation involved contributions from notable scientists like Jean le Rond D'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. The equation was first discovered in 1746 by d'Alembert in one dimension, and Euler later extended it to three dimensions. The wave equation can be derived in different physical settings, such as the vibration of a string or using Hooke's Law in the theory of elasticity.

The study of wave equations and their elementary solutions is a key focus in this unit.

10.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- Discuss about wave equation.
- Explain wave equations and its elementary solutions to wave equation.

The objectives of studying wave equations include providing a mathematical foundation to describe and understand diverse wave phenomena such as mechanical and electromagnetic waves. These equations are crucial for modeling physical systems exhibiting wave-like behavior, enabling predictions of wave propagation through different media. In the realm of elasticity, the wave equation offers insights into material deformation under stress. Studying and deriving elementary solutions contribute to a deeper understanding of fundamental wave patterns. The practical applications of wave equations in engineering involve designing and analyzing structures and systems that involve wave propagation, impacting fields like acoustics, optics, and fluid dynamics. Furthermore, technological advancements, particularly in areas like medical imaging and telecommunications, are driven by the principles elucidated by the study of wave equations. Interdisciplinary collaboration across various scientific domains is encouraged, emphasizing the broad and versatile implications of wave equations in advancing scientific knowledge and technological innovation.

10.3 *DERIVATION OF ONE DIMENSIONAL WAVE EQUATION:-*

The wave equation is a second-order linear partial differential equation that mathematically describes the behavior of waves. It expresses how a physical quantity, often represented by the variable u, varies with both time (t) and space (x) and is commonly written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here, $\frac{\partial^2 u}{\partial t^2}$represents the second partial derivative of u with respect to time, indicating the acceleration of the wave in time. $\frac{\partial^2 u}{\partial x^2}$is the second partial derivative of u with respect to space, representing the curvature of the wave in space. The variable c is the velocity of the wave.

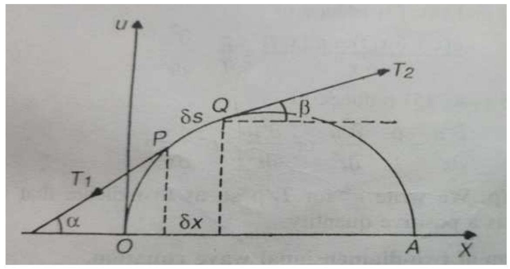

Fig.1

Let us consider an elastic string which is stretched to length fixed at end point O and A. Let that the string is distorted and further suppose that at time $t = 0$. Let OA be taken as x-axis and let a perpendicular line through O be taken as the $u - axis$.

Let we constitute the following assumption.

1. The string is homogeneous, i.e., the mass of the string per unit length is constant ρ.
2. The entre motion takes place in the $x - plane$.
3. The string is perfectly elastic and it does not produce resistance to bending.
4. The string makes small transverse vibration so that the absolute values of deflection u and the slope $\frac{\partial u}{\partial x}$ at any point of the string are small.
5. The tension in the string is large so that the force due to weight of the string could be neglected.

   Suppose the motion of the small portion PQ of length δs of the string. By hypothesis the string produces no resistance to bending so the tensions T_1 and T_2 at P and Q will act along tangent at P and Q respectively. Since there is no motion in horizontal direction, we obtain

$$T_1 cos\alpha = T_2 cos\beta = T\ (constant) \qquad \dots(1)$$

Now PQ is $\rho\delta s$ and acceleration of this element in vertical direction is $\frac{\partial^2 u}{\partial t^2}$. The resultant vertical force acting on PQ is $T_2 \sin\beta - T_1 sin\alpha$. Hence $P = mf,$ then

$$T_2 \sin\beta - T_1 sin\alpha = (\rho\delta s)\frac{\partial^2 u}{\partial t^2}$$

$$(T_2 \sin\beta)/T - \frac{T_1 sin\alpha}{T} = \frac{(\rho\delta x)}{T}\frac{\partial^2 u}{\partial t^2}, \quad \delta s = \delta x$$

$$tan\beta - tan\alpha = \frac{(\rho\delta x)}{T}\frac{\partial^2 u}{\partial t^2} \qquad \dots(2)$$

Since

$$tan\alpha = (\partial u/\partial x)_x = u(x,t)$$

$$tan\beta = (\partial u/\partial x)_{x+\delta x} = u(x+\delta x, t)$$

Putting the values of $tan\alpha$ and $tan\beta$ in equation (2), we get

$$\frac{u(x+\delta x,t) - u(x,t)}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Now putting $\delta x \to 0,$ in above equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial x^2} \quad or \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where $c^2 = T/\rho$ and is positive quantity.

10.4 *DERIVATION OF TWO DIMENSIONAL WAVE EQUATION:-*

Suppose the membrane be distorted and further let at time $t = 0$, it be released and allowed to vibrate. Let $A'B'C'D'$ be the shape of an element of the membrane at any time t.Let $ABCD$ be the projection of $A'B'C'D'$ on the $xy = plane$.

Let we constitute the following assumption.

1. The membrane is homogeneous, i.e., the mass of the membrane per unit area is constant ρ.
2. The entire motion takes place in a direction perpendicular to $xy - plane$.
3. The string is perfectly elastic and it does not produce resistance to binding.
4. The tension T per unit length developed by stretching the membrane is the same at all points and in all directions.
5. The deflection $u(x, y, z)$ is small as compared to size of the membrane. All angles of inclination are small.

6. The slopes $\partial u/\partial x$ and $\partial u/\partial y$ are small so that their higher powers can be neglected.

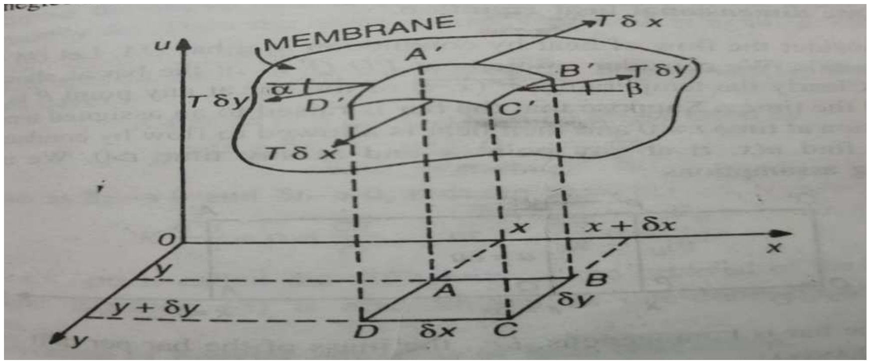

fig.2

From the above figure 2, the vertical component equal to

$$= (T\delta y)sin\beta - (T\delta y)sin\alpha$$
$$= (T\delta y)(sin\beta - sin\alpha)$$
$$= (T\delta y)(tan\beta - tan\alpha)$$

$\therefore$ α and β are so small.

$$= (T\delta y)\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

Similarly

$$= (T\delta x)\left[\left(\frac{\partial u}{\partial x}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial x}\right)_{y}\right]$$

Now the area of element $A'B'C'D'$ is $\delta x, \delta y$ so that its mass is $\rho\delta x\delta y$.
Again the acceleration in vertical direction is $\frac{\partial^2 u}{\partial t^2}$. Hence,

$$= (T\delta y)\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] += (T\delta x)\left[\left(\frac{\partial u}{\partial x}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial x}\right)_{y}\right]$$
$$= (\rho\delta x\delta y)\ \frac{\partial^2 u}{\partial t^2}$$

Or

$$\frac{u(x+\delta x, y, t) - u(x, y, t)}{\delta x} - \frac{u(x, y+\delta y, t) - u(x, y, t)}{\delta y} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Now putting $\delta x \to 0$, $\delta y \to 0$, in above equation obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Or

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2=T/\rho$. Hence which is required the two dimensional equation of wave eequation.

10.5 *DERIVATION OF THREE DIMENSIONAL WAVE EQUATION:-*

The present module focuses on solving the three-dimensional homogeneous wave equation in various coordinate systems—Cartesian, cylindrical, and spherical polar. The primary technique employed is the separation of variables. This mathematical method involves expressing the solution as a product of functions, simplifying the partial differential equation. The module explores solutions in different coordinate systems, utilizing the separation of variables approach.

Furthermore, the module delves into Duhamel's principle for addressing the inhomogeneous wave equation. Duhamel's principle provides a strategy for obtaining solutions to inhomogeneous linear partial differential equations. It involves solving the related homogeneous equation and incorporating the effects of the inhomogeneous term using an integral representation.

Solution by Separation of Variables Method:

a. **Cartesian Coordinates:** The three-dimensional wave equation in Cartesian coordinates describes the propagation of a wave in three-dimensional space. In Cartesian coordinates (x,y,z), the form of three dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Substituting $(x,y,z)=X(x),Y(y),Z(z),t(t)$, we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=\frac{1}{Tc^2}\frac{\partial^2 T}{\partial t^2}=-\frac{\omega^2}{c^2}$$

So that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-\frac{\omega^2}{c^2}=n^2$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}-n^2=-p^2$$

Thus, we have the following four ordinary differential equations to define X, Y, Z and T, viz,

$$\frac{d^2X}{dx^2}+p^2X=0, \frac{d^2Y}{dy^2}+q^2Y=0, \frac{d^2Z}{dz^2}+r^2Z=0, \frac{d^2T}{dt^2}+\omega^2T=0$$

Where $q^2=n^2-p^2, r^2=\frac{\omega^2}{c^2}-n^2$ so that $p^2+q^2+r^2=\frac{\omega^2}{c^2}$.

Thus, the complete solution of these equations can readily be acquired. Hence, the complete solution of the equation (1) is given below

$$\begin{aligned} u(x,y,z,t)=&\{c_1\, cos(px)+c_2 \sin(px)\}\{c_3\, cos(qx)\\ &+c_4 \sin(qx)\}\{c_5\, cos(rz)+c_6 sin(rz)\}\{c_7\, cos(\omega t)\\ &+c_8 sin(\omega t)\} \qquad \ldots (2)\end{aligned}$$

is periodic of period $\frac{2\pi}{\omega}$.

b. Cylindrical Polar Coordinates: In cylindrical coordinates (r,θ,z), the form of three dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial t}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Substituting $(x,y,z)=X(x),Y(y),Z(z),t(t)$, we have

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=\frac{1}{Tc^2}\frac{\partial^2 T}{\partial t^2}=-\frac{\omega^2}{c^2}$$

So that

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-\frac{\omega^2}{c^2}=-n^2$$

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+n^2r^2=-\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=m^2$$

Thus, we have the following four ordinary differential equations to define R, Θ, Z and T, viz,

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}+\left(n^2-\frac{m^2}{r^2}\right)R=0, \frac{\partial^2\Theta}{\partial\theta^2}+m^2\Theta=0,$$
$$\frac{\partial^2Z}{\partial z^2}+q^2Z=0, \frac{d^2T}{dt^2}+\omega^2T=0$$

Where $r^2=\frac{\omega^2}{c^2}-n^2$ and m assume is supposed to be a positive integer. Then the complete solution of above equation

$$\begin{aligned}u(r,\theta,z,t)=\{&c_1J_m(nr)+c_2Y_m(nr)\}\{c_3\,cos(m\theta)\\&+c_4\sin(m\theta)\}\{c_5\,cos(qz)+c_6sin(qz)\}\{c_7\,cos(\omega t)\\&+c_8sin(\omega t)\}\qquad\qquad ...(2)\end{aligned}$$

is periodic of period $\frac{2\pi}{\omega}$.

a. **Spherical Polar Coordinates:** In spherical polar coordinates (r, θ, ϕ), the form of three dimensional wave equation is
$$\frac{\partial^2u}{\partial t^2}=c^2\left(\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2sin\theta}\frac{\partial}{\partial\theta}(sin\theta)\,\frac{\partial u}{\partial\theta}+\frac{1}{r^2sin^2\theta}\frac{\partial^2u}{\partial\phi^2}\right)$$

Substituting $(r,\theta,\phi,t)=R(r),\Theta(\theta),\emptyset(\phi),t(t)$, we have

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\text{sin}\theta\Theta}\frac{d}{d\theta}(sin\theta)\left(sin\theta\frac{d\Theta}{\text{d}\theta}\right)+\frac{1}{\emptyset}\frac{1}{r^2sin^2\theta}\frac{\partial^2\emptyset}{\partial\phi^2}$$
$$=\frac{1}{Tc^2}\frac{\partial^2T}{\partial t^2}=-\frac{\omega^2}{c^2}$$

So that

$$\frac{r^2sin^2\theta}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}+\frac{\omega^2}{c^2}R\right)+\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{\text{d}\theta}\right)$$
$$=-\frac{1}{\emptyset}\frac{\partial^2\emptyset}{\partial\phi^2}-\frac{\omega^2}{c^2}=m^2$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}+\frac{\omega^2}{c^2}R\right)=-\frac{1}{\Theta sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{\text{d}\theta}\right)+\frac{m^2}{sin^2\theta}$$

$$=n(n+1)$$

Thus, we have the following four ordinary differential equations to define $R, \Theta, \emptyset$ and T, viz,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{\partial R}{\partial r}+\left(\frac{\omega^2}{c^2}R+\frac{n(n+1)}{r^2}\right)R = 0, \frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+\left\{n(n+1)-\frac{m^2}{sin^2\theta}\right\}\Theta = 0,$$
$$\frac{\partial^2\emptyset}{\partial\phi^2}+m^2\emptyset = 0, \frac{d^2T}{dt^2}+\omega^2T = 0$$

Where $r^2 = \frac{\omega^2}{c^2} - n^2$ and m assume is supposed to be a positive integer. Then the complete solution of above equation

$$u(r,\theta,z,t) = \left\{c_1J_{n+\frac{1}{2}}(\mu r)+c_2Y_{n+\frac{1}{2}}(\mu r)\right\}\{c_3\,P_n^m cos\,\theta + c_4Q_n^m cos\theta\}\{c_5\,cos(m\phi)+c_6 sin(m\phi)\}\{c_7\,cos(\omega t) + c_8 sin(\omega t)\} \qquad \ldots(2)$$

is periodic of period $\frac{2\pi}{\omega}$.

10.6 *D'ALEMBERT'S SOLUTION OF WAVE EQUATION:-*

Let the given wave equation is

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) \qquad \ldots(1)$$

Suppose u and v be two independent variables such that

$$u = x + ct \quad \text{and} \quad v = x - ct \qquad \ldots(2)$$

From (2), we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial v}$$
$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial v}$$

So that

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \dots (3)$$

From(3), we get

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}\right)$$

Again,

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial t} = c\frac{\partial \phi}{\partial u} - c\frac{\partial \phi}{\partial v}$$

$$\frac{\partial \phi}{\partial t} = c\left(\frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial v}\right)$$

So

$$\frac{\partial}{\partial x} \equiv c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial v}\right)$$

$$\frac{1}{c^2}\left(\frac{\partial^2 \phi}{\partial t^2}\right) = \frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}$$

Putting the value of $\frac{1}{c^2}\left(\frac{\partial^2 \phi}{\partial t^2}\right)$ and $\frac{\partial^2 \phi}{\partial x^2}$ reduces to (1), we obtain

$$\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2} = \frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}$$

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0$$

Integrating it, we have

$$\frac{\partial \phi}{\partial u} = F(u) \qquad \dots (3)$$

Where $F(u)$ is an arbitrary function of u.

Integrating (3) w.r.t. u, we get

$$\phi = \int F(u)\, du + g(v) = f(u) + g(v) = f(x + ct) + g(x - ct)$$

SOLVED EXAMPLE

EXAMPLE1: Obtain the general solution of wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

SOLUTION: Given

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \dots(1)$$

Let the solution of (1) be the form

$$u(x,t) = X(x)T(t) \qquad \dots(2)$$

From (1) and (2), we get

$$X''t - \frac{1}{c^2}XT'' \quad or \quad \frac{X''}{X} = \frac{T''}{c^2T}$$

$$\therefore \qquad X'' - \mu X = 0$$

$$T'' - c^2\mu T = 0$$

Solving the above equation, we have

i. when $\mu = 0$, then $X = a_1x + a_2, X = a_3t + a_4$
ii. when μ is negative sign and $-\lambda^2$, then

$$X = b_1e^{\lambda x} + b_2e^{-\lambda x}, T = b_3e^{c\lambda t} + b_4e^{-c\lambda t}$$

iii. when μ is positive sign and λ^2, then

$$X = c_1cos\lambda x + c_2sin\lambda x, T = c_3coscpt + c_4sincpt$$

The various possible solution are

$$u(x,t) = (a_1x + a_2)(a_3t + a_4)$$
$$u(x,t) = \left(b_1e^{\lambda x} + b_2e^{-\lambda x}\right)\left(b_3e^{c\lambda t} + b_4e^{-c\lambda t}\right)$$
$$u(x,t) = (c_1cos\lambda x + c_2sin\lambda x)(c_3coscpt + c_4sincpt)$$

EXAMPLE2: A string is stretched between two fixed points (0,0) and (1,0) and released at rest from the positions $u = \lambda sin\pi x$. Show that the formula for its subsequent displacement $u(x,t)$ is given by $u(x,t) = \lambda\cos(\pi ct)\, sin\pi x, c^2$ being diffusivity.

SOLUTION: The given

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \dots(1)$$

Since $u(0,t) = 0 \quad and \qquad u(1,t) = 0$

$$u(x,t) = \lambda sin\pi x = 0 \quad and \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$$

From the above equation, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos(n\pi ct) \sin(n\pi x)$$

where $\qquad C_n = 2\int_0^1 \lambda sin\pi x \; sinn\pi x \; dx$

It is obvious that $C_n = 0 \; for \; n = 2,3, \dots \dots but \; C_1 = \lambda \int_0^1 \lambda sin^2\pi x \;\; dx = \lambda$.

Hence $u(x,t) = \lambda \cos(\pi ct) \, sin\pi x, c^2$ being diffusivity.

EXAMPLE3:show that the deflection of vibrating string$\pi(its \; ends \; beingfixed \; and \; c^2 = 1)$, corresponding to zero initial velocity and initial deflection $F(x) = \lambda(sinx - sin2x)$.

SOLUTION: The given

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \; (as \; c^2 = 1) \qquad \dots (1)$$

Since $g(x) = 0 \quad and \qquad F(x) = \lambda(sinx - sin2x)$

Hence by (1), we get (as $D_n = 0$)

From the above equation, we obtain

$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{(n\pi ct)}{l}\frac{\sin(n\pi x)}{l} = \sum_{n=1}^{\infty} C_n \cos nt \; sinnx \; as \; c = 1 \; and \; \pi = l$

where $\qquad C_n = \frac{2}{\pi}\int_0^1 F(x)\frac{\sin(n\pi x)}{l} dx$

$$= \frac{2}{\pi}\int_0^1 \lambda(sinx - sin2x).\sin(nx)\,dx$$

$$= \frac{2\lambda}{\pi}\int_0^1 \lambda(sinx).\sin(nx)\,dx = \frac{2\lambda}{\pi}\int_0^1 \lambda(sin2x).\sin(nx)\,dx$$

It is obvious that $C_n = 0 \; for \; n = 3,4,5 \dots \dots but \; C_1 = \lambda, C_2 = \lambda$.

Hence $u(x,t) = C_1 cost\ sinx + C_2\ cos2tsin2x = \lambda(sinx - sin2x)$ being diffusivity.

SELF CHECK QUESTIONS

1. What is the one-dimensional wave equation?
2. State the boundary conditions for a string fixed at both ends.
3. What are the initial conditions for the wave equation on a string?
4. Provide D'Alembert's formula for the solution of the one-dimensional wave equation.
5. State the boundary conditions for a membrane (two-dimensional surface) vibrating at both edges.
6. Provide a general solution for the two-dimensional wave equation.

10.7 *SUMMARY:-*

In this unit we have studied the wave one dimensional equation, wave two dimensional equation, wave three dimensional equation, D' Alembert's principle of wave equation. The wave equation is a partial differential equation that describes how waves propagate through a medium. It is a fundamental equation in physics and engineering, commonly used to model various wave phenomena such as sound waves, electromagnetic waves, and mechanical waves. The wave equation is a mathematical description of how waves propagate through a medium. In its one-dimensional form, it relates the second partial derivative of the wave displacement with respect to time to the second partial derivative of the displacement with respect to space, multiplied by the square of the wave speed c. The equation, typically written as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, signifies how the curvature of the wave at a given point and time is linked to its spatial curvature and the speed at which it travels. The wave equation is fundamental in understanding wave phenomena across various scientific disciplines, including acoustics, optics, and electromagnetism.

10.8 *GLOSSARY:-*

- **Wave Equation:** A mathematical formula describing the behavior of waves as they propagate through a medium.
- $\boldsymbol{u(x,t)}$: Represents the wave displacement as a function of position x and t.

- $\frac{\partial^2 u}{\partial t^2}$: The second partial derivative of u with respect to time, representing the acceleration of the wave.
- $\nabla^2 u$: The Laplacian operator applied to u, involving second spatial derivatives. It characterizes the spatial variation of the wave.
- c:The wave speed, indicating how fast the wave propagates through the medium.
- **Propagation:**The movement or spread of a wave through a medium.
- **Medium:**The substance or material through which a wave travels (e.g., air, water, or a solid).
- **Laplacian Operator ∇^2 :** A mathematical operator that represents the sum of second partial derivatives with respect to spatial coordinates.
- **Acceleration of the Wave:** Describes how the wave changes its speed or direction over time.
- **Spatial Variation:** Refers to the changes in the wave's amplitude or shape across different positions in space.
- **One-Dimensional Wave Equation:** Describes wave propagation in a single spatial dimension (e.g., along a straight line).
- **Two-Dimensional Wave Equation:** Describes wave behavior in two spatial dimensions (e.g., on a surface).
- **Three-Dimensional Wave Equation:** Extends the wave equation to describe wave propagation in three spatial dimensions.
- **Separation of Variables:** A mathematical technique used to solve partial differential equations by assuming that the solution can be expressed as a product of functions, each dependent on only one variable.
- **Solution:** The expression or set of functions that satisfy the wave equation.
- **Partial Differential Equation (PDE):** An equation that relates partial derivatives of a multivariable function.
- **Dependent Variables:** The variables in a function that are being studied or analyzed.
- **Independent Variables:** Variables in a function that are assumed to be independent of each other.
- **Temporal Variable (t)**: The variable representing time in the context of the wave equation.
- **Spatial Variables (x, y, z) :** The variables representing spatial coordinates in the context of the wave equation.

- **Mode Shapes:** The spatial components of the solution, representing different patterns of the wave.
- **Eigenvalue:** A parameter in the solution that represents a constant scaling factor for a mode shape.
- **Separation Constant:** A constant introduced during the separation of variables process, often denoted by λ or k.
- **Boundary Conditions:** Constraints applied to the solution to match the specific conditions of the physical system.
- **Superposition Principle:** The principle that states the sum of individual solutions to a linear partial differential equation is also a solution.

These terms provide a foundation for understanding the key elements and concepts associated with the wave equation and its applications in diverse scientific fields.

10.9 *REFERENCES:-*

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10.10 *SUGGESTED READING:-*

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- file:///C:/Users/user/Desktop/1462442186E-textofChapter6Module3.pdf
- file:///C:/Users/user/Desktop/1462442213E-textofChapter6Module4.pdf

10.11 *TERMINAL QUESTIONS:-*

(TQ-1): Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ if the string of length $2a$ is originally plucked at the middle point by giving at it initial displacement d from the main position.

(TQ-2): A string is stretched between two fixed points (0,0) and $(0, l)$ and are released at the rest form the deflection given by

$$F(x) = \begin{cases} \frac{2\lambda}{l}x, 0 < x < \frac{l}{2} \\ \frac{2\lambda}{l}(l-x), \frac{l}{2} < x < l \end{cases}$$

Show that the deflection of string at any time t is given by

$u(x,t) = \frac{8\lambda}{\pi^2}\sum_{n=1}^{\infty} C_n \cos\frac{(n\pi ct)}{l} sin\frac{n\pi x}{l} \sin\frac{n\pi}{2}$, put $a = \frac{l}{2}$.

(TQ-3): Prove that

$u(x,y,t) = A_{12}cos\lambda_{12}t\ sin\pi x\ sin2\pi y = cos(\pi t\sqrt{5})\ sin\pi x\ sin2\pi y,$

the deflection of the square membrane of each side unity and $c = 1$, if the intial velocity is zero and initial deflection is $f(x,y) = Asin\pi x\ sin2\pi y$.

(TQ-4): Show that for a rectangular membrane of sides a and b vibrating with its boundaries fixed, eigen values and eigen function are given by

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad ,$$

$$u_{mn} = [A_{mn}cos(\lambda_{mn}t) + B_{mn}sin(\lambda_{mn}t)]$$
$$sin\frac{max}{a}sin\frac{n\pi y}{b}, m = 1,2,3, n = 1,2,3 \ldots.$$

(TQ-5): A string is stretched between two fixed points (0,0) and (1,0) and released at rest from the positions $u = \lambda sin\pi x$. Show that the formula for its subsequent displacement $u(x,t)$ is given by $u(x,t) = \lambda\cos(\pi ct)\ sin\pi x, c^2$ being diffusivity.

(TQ-6):show that the deflection of vibrating string $\pi(its\ ends\ being fixed\ and\ c^2 = 1)$, corresponding to zero initial velocity and initial deflection $F(x) = \lambda(sinx - sin2x)$.

10.12 *ANSWERS:-*

SELF CHECK ANSWERS

1. The one-dimensional wave equation is a partial differential equation that describes the motion of a wave along a one-dimensional medium. It is typically written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ represents the displacement of the wave as function of position x and time t, and c is the wave speed.

2. The boundary conditions for a string fixed at both ends are typically given as:

$$u(0,t) = 0,$$

This condition represents the fixed end at $x = 0$.

$$u(L,t) = 0$$

This condition represents the fixed end at $x = L$, where L is the length of the string.

3. The initial conditions for the wave equation on a string involve specifying the initial displacement and velocity of the string at each point x. Mathematically, this can be expressed as:

$$u(x,0) = f(x)$$

where $f(x)$ is the initial displacement function, and

$$\frac{\partial u}{\partial t} u(x,0) = g(x)$$

where $g(x)$ is the initial value function.

4. D'Alembert's formula for the solution of the one-dimensional wave equation is given by:

$$\phi = \int F(u)\, du + g(v) = f(u) + g(v) = f(x+ct) + g(x-ct)$$

Here, $u(x,t)$ is the displacement of the wave at position x and time $t, f(x)$ is the initial displacement function, $g(x)$ is the initial velocity function, c is the wave speed, and s is a dummy variable of integration.

5. Boundary conditions for a membrane typically include $u(x,y,t) = 0$ at the edges of the membrane, representing fixed boundaries.
6. The general solution for the two-dimensional wave equation involves a superposition of two-dimensional waves and can be complex. However, one common approach is to express it as a sum of sinusoidal functions, similar to the Fourier series, representing different spatial frequencies and their time evolution.

Unit 11: Green's Function

CONTENTS:

11.1 *INTRODUCTION:-*

Fig. George Green

Green's functions, named after the British mathematician George Green, are a powerful mathematical tool developed in the 1830s. These functions play a crucial role in solving differential equations, particularly those containing inhomogeneous terms or source terms. The methods involving Green's functions allow us to relate the solution of a differential equation to an integral operator.

George Green (14 July 1793 to 31 May 1841) was a British mathematical physicist known for his significant contributions to the field. In his work, An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," Green introduced crucial concepts that laid the groundwork for modern physics. These concepts include a theorem similar to the contemporary Green's theorem, the idea of potential functions as currently employed in physics, and the introduction of what are now known as Green's functions. Green's groundbreaking work marked the first mathematical theory of electricity and magnetism, serving as the foundation for subsequent advancements by scientists like James Clerk Maxwell and William Thomson. Green's contributions have had a lasting impact on the understanding and development of theories in electromagnetism.

11.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To provide a systematic way to find solutions to differential equations by representing the response of a system to a delta-function input.
- To solve boundary value problems in partial differential equations (PDEs).
- To find solutions to Laplace's equation in a given domain with specified boundary conditions.
- To solving the Poisson equation, which arises when there are distributed sources within the domain.
- To find solutions to the heat conduction equation under specific sinitial and boundary conditions.

11.3 *SIMPLE HOMOGENEOUS DIFFERENTIAL EQUATIONS:-*

Suppose the differential equation

$$\frac{d^2y}{dx^2} = 0$$

This can be explained very easily and we will get the solution as

$$y = Ax + B$$

which is the equation for a straight line.

Similarly consider another homogeneous equation given below

$$\frac{d^2y}{dx^2} + k^2y = 0$$

$$y = Asinkx + Bcoskx$$

Therefore, homogeneous equations can be solved using straightforward methods. But if we replace them with source terms like

$$\frac{d^2y}{dx^2} = Inx$$

$$\frac{d^2y}{dx^2} + k^2y = tanx$$

then the problem become difficult to solve.

11.4 *STURM LIOUVILLE OPERATOR:-*

The Sturm-Liouville operator is a differential operator frequently encountered in the study of second-order linear differential equations. Specifically, it is defined as follows:

$$L(y) = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0$$

For

$$\frac{d^2y}{dx^2} = 0$$

Since $p(x) = 1$ and $q(x) = 0$ and for

$$\frac{d^2y}{dx^2} + k^2y = 0$$

$p(x) = 1$ and $q(x) = k^2$ and any differential operator can be replaced into Sturm Liouville operator form.

11.5 *DIRAC DELTA FUNCTION:-*

The Dirac delta function, denoted as δ(x), is a mathematical function that is often used in engineering and physics to model idealized impulses or concentration of mass or charge at a single point. It was introduced by the physicist Paul Dirac. The Dirac delta function is not a conventional function in the traditional sense; rather, it is a distribution or generalized function.The Dirac delta function is defined by its behavior under integration. Formally, for any well-behaved function f(x), the defining property of the Dirac delta function is given by:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

This means that the integral of the Dirac delta function over the entire real line is equal to 1.

The Dirac delta function also has the property known as the sifting property:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = 1$$

Here, a is a constant, and the integral "sifts out" the value of the function $f(x)$ at the point $x = a$.

It's important to note that the Dirac delta function is not a conventional function with a graph. Instead, it is a mathematical tool used to represent idealized impulses or concentration of values at specific points. The Dirac delta function plays a crucial role in signal processing, control theory, quantum mechanics, and other fields where the modeling of impulses or concentrated forces is essential.

11.6 *GREEN'S FUNCTION:-*

Green's function is a mathematical concept used in the context of linear differential equations, especially in the study of boundary value problems. Given a linear, inhomogeneous differential equation of the form:

$$L[(u)X] = f(X) \qquad \dots (1)$$

where $X = (x, y, z)$ is a vector in three (or higher) dimensions and L is a linear differential operator, X with constant coefficients a $u(X)$ is the unknown function, and $f(X)$ is a given function, the Green's function $G(X, \xi)$ for the differential operator L is defined as the solution to the equation:

$$L\, G(X, \xi) = \delta(X - \xi) \qquad \dots (2)$$

Here, $\delta(X - \xi)$ is the Dirac delta function, and $\xi(x, y, z)$ is a parameter. The Green's function is often associated with a specific set of boundary conditions.

Multiplying (2) by $f(\xi)$ and integrating over the volume V of $\xi -$ space so that $dV = d\xi\, d\eta\, d\zeta$., we can be written as

$$\int_V L\, G(X, \xi) f(\xi) dV = \int_V \delta(X - \xi) f(\xi) dV = f(X)$$

Or

$$L\left\{\int_V G(X, \xi) f(\xi) dV\right\} = f(X) \qquad \dots (3)$$

Comparing the equations (1) and (3), the solution of (3) in the given form

$$u(X) = \int_V G(X, \xi) f(\xi) dV \qquad \dots (4)$$

Hence, the equation (4) is valid for any finite number of components of X.

Green's functions are particularly useful in solving to any linear, constant coefficient non homogeneous partial differential equations in any numbers of independent variables and boundary value problems. They offer a systematic method for finding solutions by breaking down a complex problem into simpler problems with localized source terms.

Generally, a Green's function is an integral kernel that can be used to explain the differential equations from a large number of families including simpler examples such as ordinary differential equations with initial or boundary value conditions, as well as more difficult examples such as inhomogeneous partial dssifferential equations (PDE) with boundary conditions.

Now another approach the problem is to look for the operator L^{-1} provided it exists. The "inverse operator" of L, denoted as L^{-1}, could be considered an operator that, when applied to the Green's function. The inverse operator can be written as an integral operator of the form

$$u(X) = L^{-1}\{f(X)\} = \int_V G(X, \xi) f(\xi) dV$$

where $G(X,\xi)$ is known as the green function and the operator l for any number of independent variables.

11.7 *ONE DIMENSIONALGREEN'S FUNCTION AND ITS PROPERTIES:-*

Let

$$L\,G(X,\xi) = \delta(X-\xi)$$

Now taking the SL(Sturm Liouville) operator is

$$\frac{d}{dx}\left(p(X)\frac{d}{dx}G(X,\xi)\right) + q(X)G(X,\xi) = \delta(X-\xi)$$

Integrating over $X \;\Rightarrow\; X{-}\in \; to \; X{+}\in$

$$\int_{X-\in}^{X+\in}\frac{d}{dx}\left(p(X)\frac{d}{dx}G(X,\xi)\right)dX + \int_{X-\in}^{X+\in}\big(q(X)G(X,\xi)\big)dX$$

$$= \int_{X-\in}^{X+\in}\delta(X-\xi)dX$$

where $\in$ is constant. Hence taking second part is zero, we get

$$p(X{+}\in)\left(\frac{dG}{dx_{X+\in}} - \frac{dG}{dx_{X-\in}}\right) = 1$$

Since the limit $\in\rightarrow 0$, we have

$$p(X{+}\in)\frac{dG}{dx_{X+\in}} - p(X{-}\in)\frac{dG}{dx_{X-\in}} = 1$$

$$\frac{dG_2}{dX} - \frac{dG_1}{dX} = \frac{1}{p(X)}$$

This property proves that the values of Green Function must be different for X less than $\in$ an dX greater than $\in$. So let the Green Function before $\in$ as $G_1(X,\xi)$ and Green Function after t as $G_1(X,\xi)$. We had taken the second integral as zero which means that

$$G_2(X, X{+}\in) - G_1(X, X{-}\in) = 0$$

At $X = \in, G_2 = G_1$ so the Green Function is

1. Continuous at boundary
2. Derivatives of the Green function are discontinuous.

These are the two properties of one dimensional Green's function.

11.8 *FORMS OF GREEN'S FUNCTION:-*

Now is to find G_1 and G_2, we have

$$G_1(X,t) = C_1u_1(X)$$

$$G_2(X,t) = C_2u_2(X)$$

where C_1 and C_2 are the function of ξ are to be irreducible. so the continuity of green function leads that

$$C_2u_2(X) - C_1u_1(X) = 0 \qquad \ldots (1)$$

Discontinuity of Green function is obtained that

$$C_2u'_2(X) - C_1u'_1(X) = -\frac{1}{p(X)} \qquad \ldots (2)$$

Multiplying (2) by $u'_2(X)$ and (1) by $u'_1(X)$ and subtracting then ,we get

$$C_2u_2(X)u'_1(X) - C_1u_1(X)u'_1(X) - C_2u'_1(X)u_2(X) + C_1u'_1(X)u_1(X) = \frac{u_2(X)}{p(X)} - 0$$

$$C_1u_2(X)u'_1(X) - C_1u'_1(X)u_2(X) = \frac{u_2(X)}{p(X)}$$

$$C_1(u_2u'_1 - u'_1u_2) = \frac{u_2(X)}{p(X)}$$

If $W = u_2u'_1 - u'_1u_2$(is known as Wronskian), then

$$C_1 = \frac{u_2(X)}{Wp(X)}, C_2 = \frac{u_1(X)}{Wp(X)}$$

Hence

$$G_1(X,t) = \frac{u_1(X)u_2(X)}{Wp(X)}$$

$$G_2(X,t) = \frac{u_2(X)u_1(X)}{Wp(X)}$$

Then we obtain the solutions as given below

$y(X) = \int_a^t G_1(X,t)f(t)dt + \int_t^b G_2(X,t)f(t)dt$ is required solution.

SOLVED EXAMPLE

EXAMPLE1: Derive the Green's function for the operator $\frac{d^2}{dx^2}$ with the boundary conditions $y(0) = 0$ and $y(1) = 0$.

SOLUTION: Let the given equation

$$\frac{d^2y}{dx^2} = f(x)$$

For the homogeneous equation

$$\frac{d^2y}{dx^2} = 0$$

$\therefore \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0, \frac{dy}{dx} = 0$ is constant. Now integrating, we get

$$y = Ax + B$$

$\Rightarrow$ First these implies $y(0) = 0, B = 0, u_1(x) = Ax, u_1(t) = At, u'_1(x) = A$

$\Rightarrow$ Second these implies $y(1) = 0, A + B = 0, B = -A, u_2(x) = Ax - A, u_2(t) = At - t, u'_2(x) = A$, then the Wronskian is

$$W = u_2u'_1 - u'_1u_2 = A^2$$

For $x < t$

$$G_1(x,t) = \frac{u_1(t)u_2(t)}{A}$$

$$G_1(x,t) = x(t-1)$$

For $x > t$

$$G_2(x,t) = \frac{u_2(t)u_1(t)}{A}$$

$$G_2(x,t) = t(x-1)$$

EXAMPLE2: Derive the Green's function for the operator$\frac{d^2}{dx^2}$ with the boundary conditions $y(0) = 0$ and $y(a) = 0$.

SOLUTION: Let the given equation

$$\frac{d^2y}{dx^2} = 0$$

$\therefore$ $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0, \frac{dy}{dx} = 0$ is constant. Now integrating, we get

$$y = Ax + B$$

$\Rightarrow$ $y(0) = 0 \Rightarrow$

$$B = 0$$

$$u_1(x) = Ax$$

$$u_1(t) = At$$

$$u'_1(x) = A$$

$\Rightarrow$ $y(a) = 0 \Rightarrow$

$$Aa + B = 0$$

$$B = -A$$

$$u_2(x) = Ax - Aa$$

$$u_2(t) = At - Ba$$

$$u'_2(x) = A$$

The Wronskian is given as below

$$W = u_2u'_1 - u'_1u_2 = A^2$$

$$= A_1xA_2 - A_1(A_2x - A_1a)$$

$$= A_1xA_2 - A_1A_2x + A_2A_1a$$

$$W = A^2a$$

For $x < t$

$$G_1(x,t) = \frac{u_1(t)u_2(t)}{A}$$

$$G_1(x,t) = \frac{x(t-1)}{a}$$

For $x > t$

$$G_2(x,t) = \frac{u_2(t)u_1(t)}{A}$$

$$G_2(x,t) = \frac{t(x-1)}{a}$$

EXAMPLE3: Derive the Green's function for the operator$\frac{d^2}{dx^2}$ with the boundary conditions $y(0) = 0$ and $y'(a) = 0$.

SOLUTION: Let the given equation

$$\frac{d^2y}{dx^2} = 0$$

So it's a solution is

$$y = Ax + B$$

The first BC obtains $y(0) = 0 \Rightarrow 0 = A \times 0 + B$, hence

$$B = 0$$

$$u_1(x) = Ax$$

$$u_1(t) = At$$

$$u'_1(x) = A$$

The second BC gives $y(a) = 0 \Rightarrow$

$$A = 0$$

$$u_1(x) = B$$

$$u_1(t) = B$$

The Wronskian is given as below

$$W = u_2u'_1 - u'_1u_2 = A^2$$

$$= A_1xA_2 - A_1(A_2x - A_1a)$$

$$= A_1xA_2 - A_1A_2x + A_2A_1a$$

$$W = -AB$$

For $x < t$

$$G_1(x,t) = \frac{u_1(t)u_2(t)}{Wp(t)}$$

$$G_1(x,t) = -x$$

For $x > t$

$$G_2(x,t) = \frac{u_2(t)u_1(t)}{Wp(t)}$$

$$G_2(x,t) = -t$$

EXAMPLE4: Deduce Green's function of the operator$\left(\frac{d^2y}{dx^2} + k^2\right)$ with the boundary conditions $y(0) = 0$ and $y(L) = 0$.

SOLUTION: Let the given equation

$$\frac{d^2y}{dx^2} + k^2 = f(x)$$

So it's a solution is

$$y = Asinkx + Bcoskx$$

$u_1(x)$ is defined as the value of y at applying first boundary condition and $u_2(x)$ is defined as the value of y at applying second boundary condition.

$y(0) = 0 \Rightarrow$

$$B = 0$$

$$u_1(x) = Asinkx$$

$$u_1(t) = Asinkt$$

$$u'_1(x) = Akcoskx$$

Similarly

$y(L) = 0 \Rightarrow$

$$0 = AsinkL + BcoskL$$

$$B = -\frac{AsinkL}{coskL}$$

$$u_2(x) = Asinkx - A\frac{sinkL}{coskL}coskx = A\left(\frac{sinkxcoskL - sinkLcoskx}{coskL}\right)$$

$$u_2(x) = \frac{Asink(x-L)}{coskL}$$

$$u_2(t) = \frac{Asink(t-L)}{coskL}$$

$$u'_1(t) = \frac{Akcosk(x-L)}{coskL}$$

Then the Wronskian is

$$W = \frac{A^2k}{coskL}(sinkL)$$

For $x < t$

$$G_1(x,t) = \frac{sinkxsink(t-L)}{ksinkL}$$

For $x > t$

$$G_2(x,t) = \frac{sink(x-L)sinkt}{ksinkL}$$

Hence which is required solution.

EXAMPLE5: Deduce Green's function of the operator$\left(y'' + \frac{1}{4y}\right) = f(x)$ with the boundary conditions $y(0) = \pi = 0$.

SOLUTION: Let the given equation

$$\frac{d^2y}{dx^2} + \frac{1}{4}y = 0$$
$$\frac{d^2y}{dx^2} + \left(\frac{1}{2}\right)^2 y = 0$$

So it's a solution is

$$y = Asin\frac{x}{2} + Bcos\frac{x}{2}$$

$y(0) = 0 \Rightarrow$

$$B = 0$$
$$u_1(x) = Asin\frac{x}{2}$$
$$u_1(t) = Asin\frac{t}{2}$$
$$u'_1(x) = A\frac{1}{2}cos\frac{1}{2}x$$

Similarly

$y(\pi) = 0 \Rightarrow$

$$A = 0$$
$$u_2(x) = Bcos\frac{x}{2}$$
$$u_2(t) = Bcos\frac{t}{2}$$
$$u'_2(x) = -B\frac{1}{2}sin\frac{1}{2}x$$

Then the Wronskian is

$$W = u_2u'_1 - u'_1u_2$$
$$W = -\frac{AB}{2}$$

For $x < t$

$$G_1(x,t) = -2sin\frac{1}{2}xcos\frac{t}{2}$$

For $x > t$

$$G_2(x,t) = -2cos\frac{1}{2}xsin\frac{1}{2}t$$

EXAMPLE6: Find the Green function $\left(\frac{d^2y}{dx^2} - k^2y\right) = f(x); y(\pm\infty) = 0$

SOLUTION: Now its solution is

$$y = Ae^{kx} + Be^{-kx}$$

Then the first condition $y = (+\infty) = 0 \Rightarrow$

$$0 = Ae^{\infty} + Be^{-\infty}$$

$Ae^{\infty} = 0, A = 0,$ so that $u_1(x) = Be^{-kx}, u_1(t) = Be^{-kt}, u_1(t) = -kBe^{-kt}$

The second condition $y = (-\infty) = 0 \Rightarrow$

$$0 = Ae^{-\infty} + Be^{+\infty}$$

$Ae^{\infty} = 0, A = 0,$ so that $u_2(x) = Ae^{kx}, u_2(t) = Ae^{kt}, u_2(t) = kAe^{kt}$

Then the Wronskian is

$$W = u_2 u'_1 - u'_1 u_2$$
$$W = -2kAB$$

For $x < t$

$$G_1(x,t) = \frac{e^{k(t-k)}}{2k}$$

For $x > t$

$$G_2(x,t) = -\frac{e^{-k(t-k)}}{2k}$$

11.9 *GREEN'S FUNCTION IN THREE DIMENSIONS:-*

In three dimensions, the Poisson's equation for Green's function is:

$$\nabla^2 G = \delta(\vec{r}_2 - \vec{r}_1)$$

Here , $r = (x, y, z)$ is the position vector, and $\delta(\vec{r}_2 - \vec{r}_1)$ is the Dirac delta function in three dimensions.

The solution to this equation depends on the specific geometry and boundary conditions of the problem. For simple cases, such as a point source, the Green's function can be expressed analytically. However, in more complex scenarios, numerical methods or specialized techniques may be employed to determine the Green's function.

11.10 *GREEN'S FUNCTION FOR POSSION'S EQUATION:-*

Suppose by the definition of Green Function

$$L\,y(x) = f(x)$$
$$LG(x,t) = \delta(x = \delta)$$

$$y(x) = \int G(x,t)\, f(t)dt$$

Now from the Poisson's equation is

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

Since $\nabla^2 G = \delta(\vec{r}_2 - \vec{r}_1)$ [Three dimension greens function for Poisson's equation]

The we obtain the definition

$$\phi(\vec{r}_2) = \int G((\vec{r}_1, \vec{r}_2)\frac{\rho(r_1)}{\epsilon_0} d^3 r_1$$

But from electrodynamics we have

$$\phi(\vec{r}_2) = \int \frac{\rho(r_1)}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|} d^3 r_1$$

Comparing we can be written as

$$G((\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|}$$

This is required Greens function for Poisson's equation.

11.11 *GREEN'S FUNCTION FOR LAPLACE EQUATION:-*

Suppose P be any point within the volume V and we are to Complete $u(P)$. Let $\overrightarrow{OP} = r$ and Q be another point in $V' = V - \sum$ or on the boundary S' of $V - \sum$ such that $\overrightarrow{OQ} = \xi$

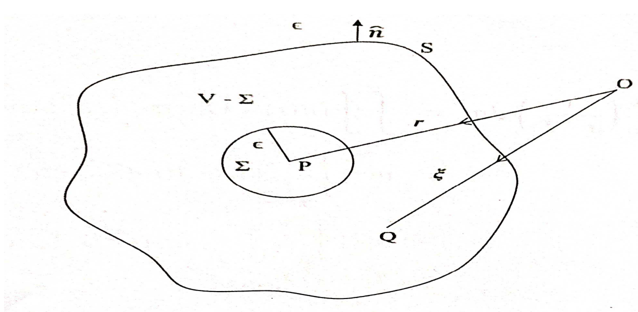

Fig. 1

From this figure So we have

$$u' = \frac{1}{|r - \xi|} \qquad \dots (1)$$

If u and u' are twice continuously differentiable functions in V, then we have

$$\iiint\limits_{V'} (u\nabla^2 u' - u'\nabla^2 u)dv = \iint\limits_{S'} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)dS' \quad \dots (2)$$

where $n\hat{}$ is the unit normal vector to dS' drawn outwards from S' and $\frac{\partial}{\partial n}$ denotes differentiation in that direction, S' being the boundary region V'.
Now $\nabla^2 u' = \nabla^2 u = 0$ within $V - \sum$, have describing (2) , we obtain
$\int\int_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)dS + \int\int_\sigma \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)d\sigma = 0$ (σ is the surface of $\sum$)
Or $\int\int_S \left[u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right) - \frac{1}{|r-\xi|}\frac{\partial u(\xi)}{\partial n}\right]ds + \int\int_\sigma \left[u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right) - \frac{1}{|r-\xi|}\frac{\partial u(\xi)}{\partial n}\right]d\sigma = 0$ ... (3)

Now if Q lies on σ , then $\frac{1}{|r-\xi|} = \frac{1}{\epsilon}$ and $\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right) = \frac{1}{\epsilon^2}$ and $d\sigma = \epsilon^2 \, sin\theta d\theta d\phi$, then

$$u(\xi) = u(r) + r.\nabla u$$

$$u(\xi) = u(r) + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= u(r) + \epsilon\left(sin\theta cos\phi\frac{\partial u}{\partial x} + sin\theta sin\phi\frac{\partial u}{\partial y} + cos\theta\frac{\partial u}{\partial z}\right)$$

$$\left[\frac{\partial u(\xi)}{\partial n} + \frac{\partial u(r)}{\partial n}\right] + O(\epsilon)$$

Again

$$\iint\limits_\sigma \left[u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right)\right]d\sigma = \iint\limits_\sigma [u(r) + O(\epsilon)].\frac{1}{\epsilon^2}\epsilon^2 \; sin\theta d\theta d\phi$$

$$= u(r)\iint\limits_\sigma \; sin\theta d\theta d\phi + O(\epsilon)$$

$$= u(r)\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} sin\theta d\theta d\phi + O(\epsilon)$$

$$= 4\pi u(r) + O(\epsilon) \qquad \dots (4)$$

and

$$\int\int_{\sigma}\left[u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right)\right]d\sigma$$

$$=\frac{1}{\epsilon}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{\partial u(\xi)}{\partial n}+O(\in)\right]\in^2\ sin\theta d\theta d\phi$$

$$=O(\in) \qquad \dots (5)$$

Putting the value of (4) and (5) in (3) ,we obtain

$$u(r)=\frac{1}{4\pi}\int\int_{S}\left[u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right)-\frac{1}{|r-\xi|}\frac{\partial u(\xi)}{\partial n}\right]ds \qquad \dots (6)$$

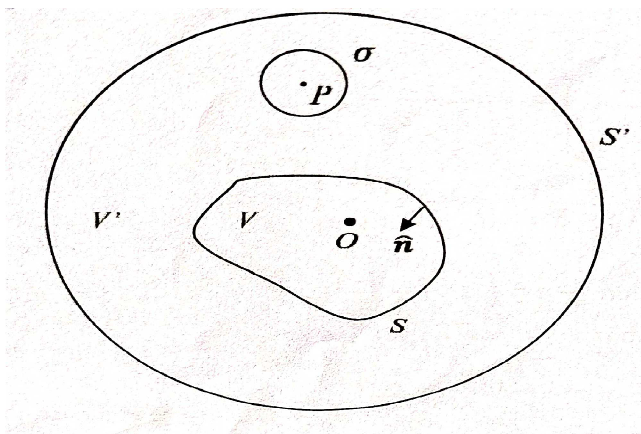

Fig.2

From the above figure , we obtain

$$4\pi u(r)+O(\epsilon)+\int\int_{S}\left[\frac{1}{|r-\xi|}\frac{\partial u(\xi)}{\partial n}-u(\xi)\frac{\partial}{\partial n}\left(\frac{1}{|r-\xi|}\right)\right]ds$$
$$+\int\int_{S'}\left[\frac{1}{R}\frac{\partial u(\xi)}{\partial n}-\frac{1}{R^2}u(\xi)\right]ds'=0$$

Now $\epsilon\to 0\ as\ R\to\infty$, we see that the (6) is also valid for exterior Derchlet problem. So the Green's equation $G(r,\xi)$ by the definition

$$G(r,\xi)=H(r,\xi)+\frac{1}{|r-\xi|}$$

Where $H(r,\xi)$ satisfied the relations

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)H(r,\xi)=0 in\ V\\ G(r,\xi)=H(r,\xi)+\frac{1}{|r-\xi|}=0\ on\ S\end{aligned}\right\} \quad \dots(7)$$

Since

$$u(r)=\frac{1}{4\pi}\int\int_S\left[G(r,\xi)\frac{\partial u(\xi)}{\partial n}-\frac{\partial G(r,\xi)}{\partial n}\right]ds \qquad \dots(8)$$

Hence it follows that if we can describe the function $G(r,\xi)$ satisfied (7) and (8) , we get

$$u(r)=-\frac{1}{4\pi}\int\int_S u(\xi)\frac{\partial G(r,\xi)}{\partial n}ds$$

is required solution.

11.12 *GREEN'S FUNCTION FOR HEAT CONDUCTION EQUATION:-*

The Green's function for the heat conduction equation is a mathematical concept used in the field of partial differential equations (PDEs), specifically in the context of heat conduction. The heat conduction equation describes how temperature changes over time and space in a given medium. This problem here is to discover the solution $u(r,t)$ of the heat conduction equation in the volume V bounded by the surface S by the use of Green's function technique

The one-dimensional heat conduction equation is typically expressed as:

$$\frac{\partial u}{\partial t}=k\nabla^2T \quad or \quad \frac{\partial u(r,t)}{\partial t}=k\frac{\partial^2u(r,t)}{\partial x^2} \qquad \dots(1)$$

where $u(x,t)$ is the temperature distribution as a function of both position x and time t, k is the thermal diffusivity of the material.

The subject to the boundary condition

$$u(r,t)=U(r,t), \qquad r\in S \qquad \dots(2)$$

and the initial condition is

$$u(r,0)=f(r), \qquad r\in V \qquad \dots(3)$$

Let us consider determine the Green's function $G(r, \xi, t-\tau), t > \tau$, where τ is a parameter satisfying the following conditions is given as

i. $\frac{\partial G}{\partial t} = k\nabla^2 G$

ii. The boundary condition is $G(r, \xi, t-\pi) = 0, \xi \in S$

iii. The initial condition is $\lim_{\tau \to t} G = 0 \ \forall \, V$ except at a point and G is singular solution of the given form

$$G(r, \xi, t-\tau) = \frac{1}{8\{\pi k(t-\tau)\}^{3/2}} exp\left\{\frac{\{r-\xi\}^2}{4k(t-\tau)}\right\} \quad \dots (4)$$

Hence $\qquad \frac{\partial G}{\partial t} = -k\nabla^2 G$

From (1) and (2), we can be written as

$$\frac{\partial u}{\partial t} = k\nabla^2 T, \quad \tau < t$$

$$u(r, \tau) = U(r, \tau), \qquad r \in S$$

Also $\qquad \frac{\partial}{\partial \tau}(uG) = G\frac{\partial u}{\partial t} + u\frac{\partial G}{\partial t} = kG\nabla^2 u - ku\nabla^2 G$

So $\int_0^{t-\epsilon}\left\{\int\int_V\int \frac{\partial}{\partial \tau}(uG)dV\right\}d\tau = k\int_0^{t-\epsilon}\left\{\int\int_V\int(G\nabla^2 u - u\nabla^2 G)dV\right\}d\tau$

Now solving the L.H.S. part of above equation

$$= \int\int_V\int\left\{\int_0^{t-\epsilon}\frac{\partial}{\partial \tau}(uG)\,dV\right\}d\tau$$

$$= \int\int_V\int\{\,(uG)_0^{t-\epsilon}d\tau\}dV$$

$$\int\int_V\int\{(uG)_{\tau=t-\epsilon} - (uG)_{\tau=0}\}\,dV \qquad \dots (5)$$

But $\{u(r,\tau)\}_{\tau=t-\epsilon} = (\xi, t-\epsilon)$. Since we can assume $(\xi, t-\epsilon) = u(r,\tau)$. Using the initial condition (3), we have

$$u(r,\tau)\int\int_V\int G(r,\xi,t-\tau)_{\tau=t-\epsilon}dV - \int\int_V\int G(r,\xi,t)\,f(\xi)dV$$

From (4), we have obtain

$$\int\int_V\int G(r,\xi,t-\tau)_{\tau=t-\epsilon}dV$$

$$=\frac{1}{8\{\pi k(\epsilon)\}^{3/2}}\int\int_V\int exp\left\{\frac{|r-\xi|^2}{4k\epsilon}\right\}dV=1, as\ \epsilon\to 0$$

Again from (10), we get

$$k\int_0^{t-\in}\{\int\int_V\int(G\nabla^2u-u\nabla^2G)dV\}\,d\tau=k\int_0^{t-\in}\left\{\int_S\int\left(G\frac{\partial u}{\partial\tau}-u\frac{\partial G}{\partial\tau}\right)dV\right\}d\tau$$

$$U(\xi,t)\frac{\partial G}{\partial n}dS$$

Since $G=0$ on S, taking limit $\epsilon\to 0$, hence the L.H.S. above equation

$$-k\int_0^t\left\{\int\int_S\int\left(G\frac{\partial u}{\partial\tau}-u\frac{\partial G}{\partial\tau}\right)dV\right\}d\tau$$

Finally ,

$$u(r,\tau)=\int\int_V\int f(\xi)G(r,\xi,t)\,dV-k\int_0^t d\tau\left\{\int\int_S\int U(\xi,t)\frac{\partial G}{\partial n}d\right\}$$

11.13 *GREEN'S FUNCTION FOR WAVE EQUATION:-*

Before describing Green's function for wave equation, we prove the following theorem, which is due to Helmholtz.

Helmholtz Theorem:

Let $u(r)$ be a function of $r=(x,y,z)$ possessing continuous partial derivatives of first and second order in a region V bounded by a closed surface S and satisfies the spatial form of the wave equation $\nabla^2u+c^2u=0\,\nabla$.

$$\frac{1}{4\pi}\int\int_V\int\left\{\frac{\exp\,(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n}-u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right)\right\}$$

$$=\begin{cases}u(r), & r\epsilon V\\ 0, & r\notin V\end{cases}\qquad\ldots(1)$$

Proof: Suppose $u(r)$ be the solution of the Helmholtz equation $\nabla^2u+c^2u=0$ in the closed region V bounded by the surface S and let all singularities of u lie outside V .So now the singularity solution is

$u'=\frac{exp(ic|r-\xi|)}{|r-\xi|}$ Green's theorem

$$\iint_V\int (u\nabla^2 u' - u'\nabla^2 u)\, dV = \iint_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS \quad \dots (2)$$

where $\hat{n}$ is outward drawn unit normal to S, we obtain of equation (2) is given by

$$\iint_V\int \left(u\nabla^2 u' - u'^{\nabla^2} u\right) dV$$

$$= \iint_V\int \left[u(\nabla^2)\left\{\frac{\exp(ic|r-\xi|)}{u(\xi)}\right\} - \frac{exp(ic|r-\xi|)}{|r-\xi|}(-c^2u)\right] dV$$

$$= \iint_V\int \left[-c^2u\left\{\frac{\exp(ic|r-\xi|)}{u(\xi)}\right\} + \frac{exp(ic|r-\xi|)}{|r-\xi|}(c^2u)\right] dV$$

$$= 0 \; (\because \; |r-\xi| \neq 0)$$

Since, from (2), we get

$$\iint_S \left\{u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right) - \frac{\exp\,(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n}\right\} dS = 0 \dots (3)$$

If P is inside V, we draw a small sphere $\sum$ with radius ϵ to contain P. Next, we apply Green's theorem to the region $V - \sum$ that is limited by S on the outside and σ on the inside. Next

$$\left(\iint_S + \iint_\sigma .\right)\left[u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right) - \frac{\exp(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n}\right] dS$$

$$= 0(\because \; |r-\xi| \neq 0) in\, V - \sum$$

So the σ

$$\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right) = \frac{\partial}{\partial \in}\left(\frac{exp(ic|r-\xi|)}{\in}\right)$$

$$= \left(ic - \frac{1}{|r-\xi|}\right)\frac{exp(ic|r-\xi|)}{|r-\xi|}$$

Again from (3) ,we obtain

$$\iint_S \left[u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right) - \frac{\exp(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n}\right] dS$$

$$= -\iint_\sigma \left[\left(ic - \frac{1}{|r-\xi|}\right)u(\xi) - \frac{\partial u(\xi)}{\partial n}\right]\frac{exp(ic|r-\xi|)}{|r-\xi|} dS \qquad \dots (4)$$

Now

$$u(\xi) = u(r) + O(\in), \qquad \frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial n}\right)_P + O(\in), dS = \in^2 \sin\theta \, d\theta d\phi$$

Again from (4), we get

$$\int\int_S \left[u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right) - \frac{\exp(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n}\right] dS$$

$$= \int\int_\sigma \left[\left(ic - \frac{1}{|r-\xi|}\right)[u(r) + O(\in)] - \frac{\partial u(\xi)}{\partial n} - O(\in)\right]\frac{exp(ic|r-\xi|)}{|r-\xi|} \in^2 \sin\theta \, d\theta d\phi$$

$$= -4\pi u(r)$$

Hence

$$\frac{1}{4\pi}\int\int_V\int \left\{\frac{\exp(ic|r-\xi|)}{u(\xi)}\frac{\partial u(\xi)}{\partial n} - u(\xi)\frac{\partial}{\partial n}\left(\frac{exp(ic|r-\xi|)}{|r-\xi|}\right)\right\}$$

$$= \begin{cases} u(r), & r \epsilon V \\ 0, & r \notin V \end{cases}$$

is required solution.

SELF CHECK QUESTIONS

1. What is a Green's function?
2. How is the Green's function used in solving differential equations?
3. What is the significance of the Green's function in boundary value problems?
4. How does the Green's function relate to the concept of impulse response in control systems?
5. What is the connection between Green's functions and eigenvalue problems?
6. How is the Green's function used to solve the Poisson equation in electrostatics?
7. Can the Green's function be used for non-linear differential equations?
8. What is the physical interpretation of the Green's function in the context of heat conduction?
9. How does the choice of boundary conditions affect the determination of the Green's function?
10. When the use of Green's is functions particularly advantageous in problem-solving?

11.14 *SUMMARY:-*

In this unit we have studied the green's function, simple homogeneous differential equations, Sturm Liouville's operator, Dirac delta function, one dimensional Green's function and its properties, forms of green's function, green's function in three dimensions, Green's function for Poisson's equation, Green's function for Laplace equation and Green's function for heat conduction equation. Green's function is a mathematical tool used in the field of partial differential equations to find solutions for a given differential equation with specific boundary or initial conditions. It is particularly valuable for linear, homogeneous differential equations. Green's function provides a powerful and systematic approach for solving linear partial differential equations by breaking down complex problems into simpler components and addressing them through integral representations. It is widely applied in various scientific and engineering disciplines to analyze and solve problems involving waves, heat transfer, electromagnetism, and other physical phenomena.

Overall, the unit covered the foundational aspects of Green's function, its application to different types of differential equations, and its role in solving problems related to heat conduction, potential theory, and boundary value problems. Students gained a comprehensive understanding of these mathematical tools and their practical applications in various physical phenomena.

11.15 *GLOSSARY:-*

Here's a glossary defining key terms related to Green's Function:

- **Green's Function:** A mathematical function used to solve differential equations, representing the response of a linear system to a point source or initial condition.
- **Differential Equation:** An equation involving derivatives that describes the relationship between a function and its derivatives. Green's function is often employed to solve linear differential equations.

- **Linear System:** A system that exhibits linearity in its response, meaning the output is directly proportional to the input. Green's function is particularly useful for linear systems.

- **Homogeneous Equation:** A differential equation in which the right-hand side is zero. Green's function is often applied to the homogeneous form of differential equations.
- **Inhomogeneous Equation:** A differential equation with a non-zero term on the right-hand side, representing an external source or forcing function. Green's function is used to find solutions to inhomogeneous equations.
- **Boundary Conditions:** Conditions specified at the boundaries of a system or domain, influencing the form of the Green's function for a particular problem.
- **Initial Conditions:** Conditions specified at the initial time or starting point of a process, influencing the form of the Green's function for problems involving time evolution.
- **Dirac Delta Function:** A mathematical function, often denoted as $\delta(x)$, representing an idealized impulse or point source. Green's function is frequently associated with the Dirac delta function in problems with point sources.
- **Convolution:** A mathematical operation that combines two functions to produce a third, expressing how one function modifies the other. Convolution is often used in the context of Green's function.
- **Integral Representation:** The expression of a solution to a differential equation as an integral involving the Green's function and the given source or initial condition.
- **Wave Equation:** A partial differential equation describing the behavior of waves. Green's function is commonly used to solve wave equations.
- **Heat Conduction Equation:** A partial differential equation describing the distribution of heat in a material over time. Green's function can be applied to solve heat conduction problems.
- **Electrostatics:** The study of stationary electric charges and their interactions. Green's function is used to solve problems related to electric potential in electrostatics.
- **Linear Operator:** An operator that satisfies the properties of linearity, playing a key role in the definition and application of Green's function.
- **Convolution Integral:** The mathematical operation used to combine the Green's function with the source or initial condition in integral representations of solutions to differential equations.

This glossary provides definitions for terms associated with Green's Function, offering clarity on concepts related to its application in solving differential equations and analyzing physical systems.

11.16 *REFERENCES:-*

- A. N. Tikhonov and A. A. Samarskii (1990), Partial Differential Equations of Mathematical Physics.
- George B. Arfken and Hans J. Weber (2005), Mathematical Methods for Physicists.

11.17 *SUGGESTED READING:-*

- Ivar Stakgold and Michael J. Holst (2011),Green's Functions and Boundary Value Problems.
- George B. Arfken, Hans J. Weber, and Frank E. Harris (2012), Mathematical Methods for Physicists: A Comprehensive Guide.
- https://epgp.inflibnet.ac.in/Home/ViewSubject?catid=ZLCHeZEhCZ8yCri36nSF3A==
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11.18 *TERMINAL QUESTIONS:-*

(TQ-1): The Green's function for the operator$\frac{d^2}{dx^2}$ with the boundary condition $y(0) = 0$ and $y'(1) = 0$.

(TQ-2): Define Green's function in one and three dimensions.

(TQ-3): Explain how the method of Green's function is useful in obtaining the solutions of Poisson's equation.

(TQ-4): Explain the Green's function for the Heimholtz equation $(\nabla^2 + c^2)u(x, y, z = 0)$ for the half-space $z \geq 0$ and hence solve it.

(TQ-5): Find the Green's function for the Heat flow problem in a finite rod described by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 \leq x \leq L$$
$$u(x, 0) = f(x), 0 \leq x \leq L,$$
$$u(x, 0) = u(L, t) = 0, \quad t > 0.$$

(TQ-6): For the sphere of radius a and centre at the origion, show that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$ is the Dirac-delta function.

(TQ-7): Prove that Green's function $G(r, \xi)$ has the symmetric property.

11.19 *ANSWERS:-*

SELF CHECK ANSWERS

1. A Green's function is a mathematical tool used in solving differential equations. For a linear, inhomogeneous differential equation, the Green's function represents the response of the system to an impulse function at a specific point.
2. The Green's function is used to obtain the particular solution to an inhomogeneous differential equation by convolving it with the forcing function. It allows us to express the solution as a weighted sum of the Green's function evaluated at different points.
3. In boundary value problems, the Green's function provides a method for solving the problem by converting it into an integral equation. It simplifies the solution process by breaking down the problem into simpler, localized contributions.
4. The Green's function is analogous to the impulse response in control systems. It describes how a system responds to an impulse input, and the convolution of the Green's function with a given input represents the system's response to that input.
5. In some cases, solving eigenvalue problems is related to finding the eigenfunctions and eigenvalues of the Green's function. This connection is particularly evident in problems involving partial differential equations.
6. In electrostatics, the Green's function can be used to solve the Poisson equation by representing the charge distribution as a distribution of point charges. The solution is then expressed as an integral involving the Green's function and the charge distribution.
7. The traditional Green's function approach is primarily applicable to linear differential equations. For non-linear equations, alternative methods, such as perturbation techniques or numerical methods, are often employed.
8. In heat conduction problems, the Green's function represents the temperature distribution resulting from a localized heat source. It provides insights into how the system responds to a sudden change in temperature at a specific point.
9. The choice of boundary conditions is crucial in determining the Green's function. Different boundary conditions lead to different

Green's functions, and selecting appropriate conditions is essential for obtaining meaningful solutions to specific problems.

10. Green's functions are particularly advantageous in problems where the boundary conditions are well-defined, and the system can be decomposed into localized responses. They provide a systematic and efficient approach to solving differential equations in such cases.

BLOCK IV
NUMERICAL SOLUTION OF PDEs

Unit 12: Finite differences of PDEs

CONTENTS:

12.1 *INTRODUCTION:-*

Most of branches branches of applied mathematics, such as fluid dynamics, boundary layer flow, elasticity, heat transfer, and electromagnetic theory, commonly deal with problems formulated as partial differential equations (PDEs). However, the complexity of these problems often limits the applicability of analytic methods, and only a small subset can be solved analytically. The solutions obtained through analytic methods are typically intricate and demand advanced mathematical techniques. Careful examination of the mathematical structures of PDEs reveals numerous challenges in demonstrating solutions for these problems. As a result, numerical methods, particularly the finite difference method, emerge as practical tools for approximating solutions to PDEs in these diverse applied fields.simple and efficient numerical methods provide a practical approach to obtaining sufficiently approximate solutions for partial differential equations (PDEs). While there are various numerical methods available, the finite difference method stands out as a popular and widely used technique. In this method, partial derivatives and boundary conditions in the PDE are replaced by their finite difference approximations. This transformation results in converting the given PDE into a system of linear equations. Despite the iterative methods used for solving these linear equations being relatively slow, they consistently yield good results in approximating solutions to complex PDE problems.

In this unit focuses on addressing the solution of Laplace's equation and introduces numerical methods—Schmidt method, Crank-Nicolson method, and Frankel method. These methods are specifically designed for solving one-dimensional heat equations and wave equations. The content delves into the application of these techniques to efficiently compute solutions for mathematical problems related to heat propagation and wave behavior in one-dimensional scenarios.

12.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To discuss the classification Second order of PDEs.
- To solve the finite difference approximations to partial derivatives.
- To solving the notation for functions of several variables.
- To discuss about the solution of Laplace equation, Heat equation, wave equation.

12.3 *CLASSIFICATION OF SECOND ORDER OF PARTIAL DIFFERENTIAL EQUATIONS:*

Let us consider the general second order linear partial differential equation in two independent variable of the form is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad \dots (1)$$

This can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad \dots (2)$$

where A, B, C are constant and the functions of x and y.
Now equating from (1), we have

$$\Delta^2 = B^2 - 4AC$$

Thus the equation (1) is called:

i. **Elliptic if** $\Delta < 0$ *i.e.*, $B^2 - 4AC < 0$
ii. **Parabolic if** $\Delta = 0$ *i.e.*, $B^2 - 4AC - 0$
iii. **Hyperbolic if** $\Delta > 0$ *i.e.*, $B^2 - 4AC > 0$

In the following, we restrict our-self to three simple particular cases of equation (1) can be written as

i. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *or* $u_{xx} + u_{yy} = 0$: **Laplace Equation**
ii. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ *or* $u_{xx} - u_{yy} = 0$: **Wave Equation**
iii. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ *or* $u_{xx} = u_t$: **Heat Equation**

If we categorize these equations, we describe that the Laplace equation is ***elliptic*** whereas the wave equation is ***hyperbolic*** and Heat equation is ***paraboli***c respectively.

In the study of partial differential equations, usually three types of problems arise:

(i) **Dirchlet's Problem:** Finding a function $u(x, y)$ that satisfies the Laplace equation in region R is necessary given a continuous function f on its boundary C, That is, finding $u(x, y)$ such that

$$\left.\begin{aligned} u_{xx} + u_{yy} = 0 \ in\ R \\ u = f \ on\ C \end{aligned}\right\} \qquad \dots (3)$$

(ii) **Cauchy problem:**

$$\left.\begin{aligned} u_{tt} - u_{xx} = 0 for\ t > 0 \\ u(x,0) = f(x) \\ \frac{\partial u(x,0)}{\partial t} = g(x) \end{aligned}\right\} \quad \ldots (4)$$

(iii) $$\left.\begin{aligned} u_{xx} = u_t\ for\ t > 0 \\ u(x,0) = f(x) \end{aligned}\right\} \quad \ldots (5)$$

In partial differential equations, the form of the equation is always connected with a particular type of boundary conditions. In this case, the problem is called **well defined or well-posed.** The problems defined in equation (3) to (5) are well –posed. If we connect Laplace equation with Cauchy boundary conditions, the problem is called **ill-posed**. Thus we have

$$\left.\begin{aligned} u_{xx} + u_{yy} = 0 \\ u(x,0) = f(x) \\ u_y(x,0) = g(x) \end{aligned}\right\} \quad \ldots (6)$$ is an *ill-posed*.

12.4 *FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES:-*

Let x and y are two independent variable of the obtained partial differential equation, so the $xy-$plane be divide into set of equal rectangles of lengths $\Delta x = h$ and $\Delta y = k$, we get

$$x = ih, \quad i = \pm 0, \pm 1, \pm 2, \ldots \ldots$$

$$y = jk, \quad j = \pm 0, \pm 1, \pm 2, \ldots \ldots$$

The points of interaction of horizontal and vertical lines are known as mesh point, lattice points or grid points. The jth mesh point is denoted by $P(x_i, y_j)$ or $P(ih, jk)$. The value of u at this mesh point is denoted by $u_{i,j}, i.e.,$ $u_{i,j} = u(x_i, y_j) = u(ih, jk)$.

$u_x(ih, jk) = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h)$, (forward difference approximation)

$u_x(ih, jk) = \frac{u_{i,j} - u_{i-1,j}}{h} + O(h)$, (backward difference approximation)

$u_x(ih, jk) = \frac{u_{i+1,j} - u_{i-1,j}}{h} + O(h^2)$, (central difference approximation)

Similarly,

$u_x(ih, jk) = \frac{u_{i,j+1} - u_{i,j}}{h} + O(k)$, (forward difference approximation)

$u_x(ih, jk) = \frac{u_{i,j} - u_{ij+1}}{h} + O(k)$, (backward difference approximation)

$u_x(ih, jk) = \frac{u_{i,j} - u_{i,j-1}}{h} + O(k^2)$, (central difference approximation)

The second order partial differential derivatives are approximated as follows:

$u_{xx}(ih, jk) = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2)$,(central difference approximation)

$u_{yy}(ih, jk) = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} + O(k^2)$ (central difference approximation)

Hence the above equations used to approximate a partial differential equation to a system of difference equations.

12.5 *ELLIPTIC EQUATION:-*

The elliptic equation is a type of partial differential equation (PDE) that often arises in physics and engineering, particularly in problems involving steady-state conditions or equilibrium. The general form of a most elliptic equation, often referred to as the Poisson equation, is obtained by:

$$\nabla^n u = 0$$

and Poisson's equation is

$$\nabla^n u = f(r)$$

The Poisson equation is a special case of the more general elliptic equation. Elliptic equations are characterized by their elliptic, positive-definite operators, and they appear in various physical phenomena, such as heat conduction, electrostatics, fluid flow, and structural mechanics. The solution to the elliptic equation represents the distribution of the unknown field u in the given domain, subject to the specified boundary conditions. The boundary conditions are essential for determining a unique solution to the problem.

12.6 *METHOD OF FIRST APPROXIMATION VALUE OF LAPLACE EQUATION:-*

In previous unit we have already studied of Laplace's equation. Let the Laplace equation in two dimension is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \ or \ u_{xx}+u_{yy}=0 \quad ...(1)$$

Suppose a rectangular region R for which $u(x,y)$ is called the boundary.

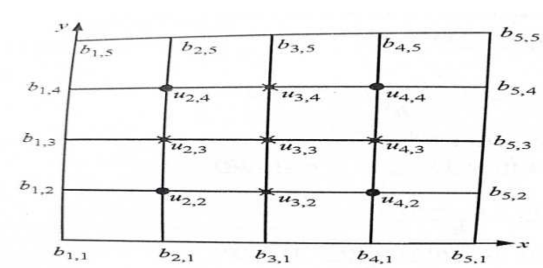

Fig.1

From figure, changing the derivatives in (1) by their difference approximations, we obtain,

$$\frac{1}{h^2}\left[u_{i-1,j}-u_{i,j}+u_{i+1,j}\right]+\frac{1}{h^2}\left[u_{i,j-1}-u_{i,j}+u_{i,j+1}\right]=0$$

Or

$$u_{i,j}=\frac{1}{4}\left[u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}\right] \quad ...(2)$$

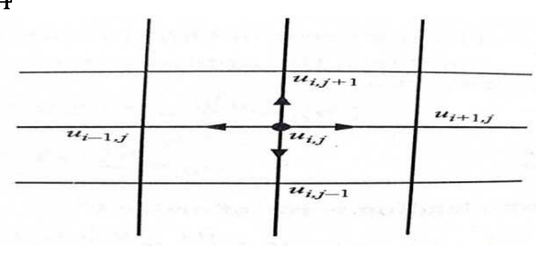

Fig.2

Hence, the equation (2) s called **Standard 5-point formula**. This formula is presented in figure 2.

Sometimes a formula similar to (2) is used, which is obtained by

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right] \quad \dots (3)$$

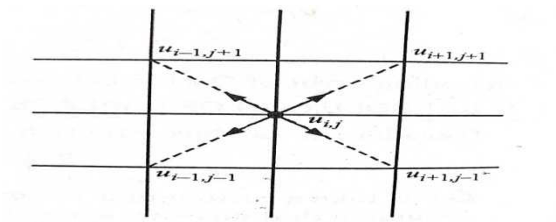

Fig.3

This formula obtains the value of $u_{i,j}$ which is the average of its values at the four neighboring mesh points at the end points of diagonal of the square. So that the formula (3) is called **Diagonal 5-point formula** is shown as figure 3.

Let we first use the diagonal 5-points formula (3) to find the initial values of u at the interior mesh points and compute $u_{3,3,}, u_{2,4}, u_{4,4}, u_{4,2}$ and $u_{2,2}$ in this order given below

$$u_{3,3,} = \frac{1}{4}[b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}]$$

$$u_{2,4} = \frac{1}{4}[b_{1,5} + u_{3,3} + u_{3,5} + u_{1,3}]$$

$$u_{4,4} = \frac{1}{4}[b_{5,5} + b_{5,3} + b_{3,5} + u_{3,3}]$$

$$u_{4,2} = \frac{1}{4}[u_{3,3} + b_{5,3} + b_{5,1} + b_{3,1}]$$

$$u_{2,2} = \frac{1}{4}[b_{1,3} + u_{3,3} + b_{3,1} + b_{1,1}]$$

Now we compute the values of u at the remaining interior points $u_{2,3,}, u_{3,4}, u_{4,3}, u_{3,2}$ by standard 5-points formula (2), we obtain

$$u_{2,3,} = \frac{1}{4}[b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}]$$

$$u_{3,4} = \frac{1}{4}[u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3}]$$

$$u_{4,3} = \frac{1}{4}[u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}]$$

$$u_{3,2} = \frac{1}{4}[u_{3,3} + u_{2,2} + b_{3,1} + u_{4,2}]$$

After writing all the nine values of $u_{i,j}$ once, their accuracy is improved by using either of the following iterative methods.

a. **Iterative method:** if initial approximations of the variable u are known, they can be refined or updated using established iterative methods. Numerous iterative techniques exist, each with varying rates of convergence. The text implies a focus on discussing several of some methods and their respective convergence properties.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 g_{i,j}\right]$$

b. **Jacobi's method:** This method obtains

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i\,,j}^{(n)} + u_{i,j-1}^{(n)} - h^2 g_{i,j}]$$

where $u_{i\,,j}^{(n)}$ is nth approximate value of $u_{i,j}$.

c. **Gauss-Seidel method:** The Gauss-Seidel method is an iterative numerical technique used to solve a system of linear equations. It is particularly useful for solving systems with a large number of equations. This method gives

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i\,,j+1}^{(n+1)} + u_{i,j-1}^{(n)} - h^2 g_{i,j}]$$

d. **Successive Over-Relaxation (SOR) method:** In this method, the iteration scheme is accelerated by introducing a scalar, called re-laxation factor. This acceleration is made by making corrections on $\left[u_{i,j}^{(n+1)} - u_{i\,,j}^{(n)}\right]$. Suppose $u_{i,j}^{\overline{(n+1)}}$ is the value given from any iteration method, such as Jacobi's or Gauss-

Seidel's method. Then the updated value of $u_{i,j}$ at the $(r+1)th$ iteration is obtain by

$$u_{i,j}^{(n+1)} = \overline{u_{i,j}^{(n+1)}}\omega - (1-\omega)u_{i,j}^{(n)}$$

where w is called relaxation factor.
If $w > 1$ then the method is known as over-relaxation method.
If $w = 1$ then the method is nothing but the Gauss-Seidal iteration method.
Hence, for the Poisson's equation, the Jacobi's over-relaxation scheme is given as

$$u_{i,j}^{(n+1)} = \frac{1}{4}w\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)} - h^2 g_{i,j}\right] + (1 - w)u_{i,j}^{(n)}$$

and the Gauss-Seidel's over-relaxation scheme is obtain as below

$$u_{i,j}^{(n+1)} = \frac{1}{4}w[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)} - h^2 g_{i,j}] + (1 - w)u_{i,j}^{(n)}$$

12.7 *SOLUTION OF POISSON'S EQUATION:-*

Its method of solution is similar to that of the Laplace equation. Here the standard five-point formula for (1) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \dots(1)$$

$$\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] = h^2 g_{i,j}\, f(x,y) \quad \dots(2)$$

By putting (2) at each interior mesh point, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by the Gauss-Seidal method.

SOLVED EXAMPLE

EXAMPLE1: Solve Laplace's equation for the square region shown in figure the boundary values being as indicated.

SOLUTION: In this figure that the boundary values are symmetric about AC. Hence $u_1 = u_4$ and find u_1, u_2, u_3.So that

$$u_2 + u_3 + 2 + 4 - 4u_1 = 0$$

Hence

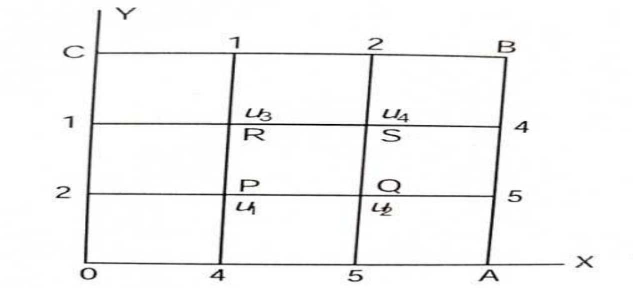

Fig.4

$$u_1 = \frac{1}{4}(u_2 + u_3 + 6)$$

The iteration formula is given as

$$u_1^{(n+1)} = \frac{1}{4}(u_2 + u_3 + 6)$$

Similarly

$$u_2^{(n+1)} = \frac{1}{2}u_1^{(n+1)} + \frac{5}{2}$$

and

$$u_3^{(n+1)} = \frac{1}{2}u_1^{(n+1)} + \frac{1}{2}$$

For the first iteration, suppose $u_2 = 5$, $u_3^{(0)} = 1$. Hence

$$u_1^{(1)} = \frac{1}{4}(5 + 1 + 6) = 3$$

$$u_2^{(1)} = \frac{1}{2}(3) + \frac{5}{2} = 4$$

$$u_3^{(1)} = \frac{1}{2}(3) + \frac{1}{2} = 2$$

For the second iteration, we obtain

$$u_1^{(2)} = \frac{1}{4}(4 + 2 + 6) = 3$$

$$u_2^{(2)} = \frac{1}{2}(3) + \frac{5}{2} = 4$$
$$u_3^{(2)} = \frac{1}{2}(3) + \frac{1}{2} = 2$$

Since the values are unchanged, we deduce that $u_1 = 3, u_2$=4, $u_3 = 2$ and $u_4 = 3$.

EXAMPLE2: Solve the elliptic equation $u_{xx} + u_{yy} = 0$ (Laplace equation) for the following square mesh with boundary values as shown.

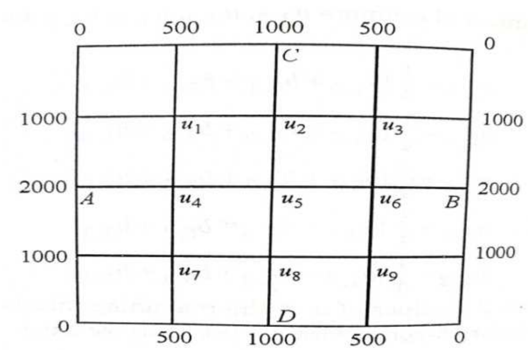

Fig.5

SOLUTION: From the above figure, it is clear that u_1, u_2, u_3 … … ….9 values of u at the interior mesh points.

Since

$$u_7 = u_1, u_8 = u_2, u_9 = u_3$$
$$u_3 = u_1, u_6 = u_4, u_9 = u_7.$$

Therefore, by standard 5-points formula

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$$

by diagonal 5-points formula

$$u_1 = \frac{1}{4}(0 + 1000 + u_5 + 2000)$$
$$u_1 = \frac{1}{4}(0 + 1000 + 1500 + 2000) = 1125$$

Again by standard 5-points formula

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1128$$
$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1438$$

Now we use the following formule

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^n + 500 + u_4^n]$$
$$u_2^{(n+1)} = \frac{1}{4}[u_1^{n+1} + u_1^n + 1000 + u_5^n]$$
$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^n + u_1^{n+1} + u_1^n]$$
$$u_5^{(n+1)} = \frac{1}{4}[u_4^{n+1} + u_4^n + u_2^{n+1} + u_2^n]$$

First iteration:

$$u_1^1 = \frac{1}{4}[1000 + 1128 + 500 + 1438] = 1032$$
$$u_2^1 = \frac{1}{4}[1032 + 1125 + 1000 + 1500] = 1164$$
$$u_4^1 = \frac{1}{4}[2000 + 1500 + 1032 + 1125] = 1414$$
$$u_5^1 = \frac{1}{4}[1414 + 1438 + 1164 + 1138] = 1301$$

Second iteration:

$$u_1^2 = \frac{1}{4}[1000 + 1164 + 500 + 1414] = 1020$$
$$u_2^2 = \frac{1}{4}[1020 + 1032 + 1000 + 1301] = 1088$$
$$u_4^2 = \frac{1}{4}[2000 + 1301 + 1020 + 1032] = 1338$$
$$u_5^2 = \frac{1}{4}[1338 + 1414 + 1088 + 1164] = 1251$$

Third iteration:

$$u_1^3 = \frac{1}{4}[1000 + 1088 + 500 + 1338] = 982$$
$$u_2^3 = \frac{1}{4}[982 + 1020 + 1000 + 1251] = 1088$$
$$u_4^3 = \frac{1}{4}[2000 + 1251 + 982 + 1020] = 1313$$
$$u_5^2 = \frac{1}{4}[1313 + 1338 + 1063 + 1088] = 1201$$

Forth iteration:

$$u_1^4 = \frac{1}{4}[1000 + 1063 + 500 + 1313] = 969$$
$$u_2^4 = \frac{1}{4}[969 + 982 + 1000 + 1201] = 1038$$
$$u_4^4 = \frac{1}{4}[2000 + 1201 + 969 + 982] = 1288$$

$$u_5^4 = \frac{1}{4}[1288 + 1313 + 1038 + 1063] = 1176$$

Similarly

$$u_1^5 = 957,\quad u_2^5 = 1026,\quad u_4^5 = 1276,\quad u_5^5 = 1157$$
$$u_1^6 = 951,\quad u_2^6 = 1016,\quad u_4^6 = 1266,\quad u_5^6 = 1146$$
$$u_1^7 = 946,\quad u_2^7 = 1011,\quad u_4^7 = 1260,\quad u_5^7 = 1138$$
$$u_1^8 = 943,\quad u_2^8 = 1007,\quad u_4^8 = 1257,\quad u_5^8 = 1134$$
$$u_1^9 = 941,\quad u_2^9 = 1005,\quad u_4^9 = 1255,\quad u_5^9 = 1131$$
$$u_1^{10} = 940,\quad u_2^{10} = 1003,\quad u_4^{10} = 1252,\quad u_5^{10} = 1129$$
$$u_1^{11} = 939,\quad u_2^{11} = 1002,\quad u_4^{11} = 1252,\quad u_5^{11} = 1128$$
$$u_1^{12} = 939,\quad u_2^{12} = 1001,\quad u_4^{12} = 1251,\quad u_5^{12} = 1126$$

Here 11th and 12th iteration is very close

So

$u_1 = 939\,, u_2 = 1001\,, u_4 = 1251, u_5 = 1126$

EXAMPLE3: Solve the equation $u_{xx} + u_{yy} = 0$ (Laplace equation) defined in the domain of figure 6 by

- **a.** Jacobi's method
- **b.** Gauss-Seidel's method
- **c.** Gauss-Seidel's successive over relaxation method.

SOLUTION:

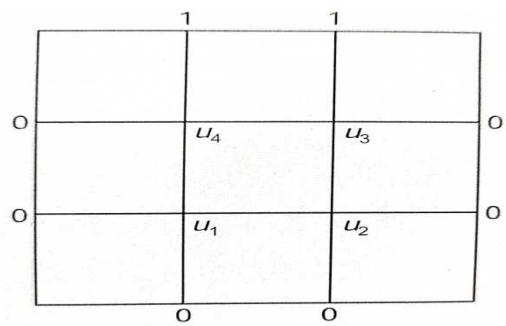

Fig.6

From the figure

- **a. Jacobi's method:** Now we investigate the approximate values of So $u_1, u_2\,, u_3, u_4$ as follows

$$u_1^1 = \frac{1}{4}[0 + 0 + 0 + 1] = 0.25$$
$$u_2^1 = \frac{1}{4}[0 + 0 + 0 + 1] = 0.25$$
$$u_3^1 = \frac{1}{4}[1 + 1 + 0 + 0] = 0.5$$

$$u_4^1 = \frac{1}{4}[1 + 1 + 0 + 0] = 0.5$$

Hence the iterations have been continued using above jacobi's method and seven successive are obtained below:

u_1	u_2	u_3	u_4
0.1875	0.1875	0.4375	0.4375
0.15625	0.15625	0.40625	0.40625
0.14062	0.14062	0.39062	0.39062
0.13281	0.13281	0.38281	0.38281
0.12891	0.12891	0.37891	0.37891
0.12695	0.12695	0.37695	0.37695
0.12598	0.12598	0.37598	0.37598

b. **Gauss-Seidel's method:** Similarly five successive iterative are obtained below:

u_1	u_2	u_3	u_4
0.25	0.3125	0.5625	0.46875
0.21875	0.17187	0.42187	0.39844
0.14844	0.13672	0.38672	0.38086
0.13086	0.12793	0.37793	0.37646
0.12646	0.12573	0.37573	0.37537

c. **Gauss-Seidel's successive over relaxation method(SOR):** With $w = 1{,}1$, three successive iterative obtained by using (SOR) are given as

u_1	u_2	u_3	u_4
0.275	0.35062	0.35062	0.35062
0.16534	0.10683	0.38183	0.37432
0.11785	0.12181	0.37216	0.37341

12.8 *PARABOLIC EQUATION:-*

A parabolic equation is a type of partial differential equation (PDE) that describes a time-dependent process. The general form of a one dimensional **Parabolic Equation**, often known as the **Heat Equation,** is given by:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \ldots (1)$$

With the initial condition $u(x, 0) = f(x)$ and the boundary condition $u(0, t) = \phi(t), u(1, t) = \psi(t)$. Here, $u(x, t)$ is the unknown function representing the distribution of some quantity (e.g., temperature) over space (x) and time(t). The coefficient α is a positive constant that determines the rate of diffusion or spread of the quantity.

12.9 *AN SCHMIDT EXPLICIT METHOD:-*

The explicit method is a numerical technique used for solving partial differential equations (PDEs) or systems of ordinary differential equations (ODEs). In the context of solving PDEs, such as the heat equation or the wave equation, the explicit method is a finite difference approach where the solution at a given time step is expressed explicitly in terms of the solution at previous time steps.

Suppose a rectangular mesh in the $x - t\ plane$ with spacing h along x direction and k along time t direction. Representing a mesh point $(x, t) = (hi, jk)$ as simply i, j, we obtain

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \approx kc^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Where $x_i = ih$ and $t_j = jk; j = 0,1,2, \dots ..$

From above equation

$$u_{i,j+1} = \alpha[u_{i-1,j} - (1 - 2\alpha)u_{i,j} + u_{i+1,j}] \quad \dots (2)$$

where $\alpha = \frac{kc^2}{h^2}$.Thus the equation (2) is a relation between the function values at the two time levels $j + 1$ and j. Hence it is called as **2-Level Formula**.The Schematic form as shown in figure 7.

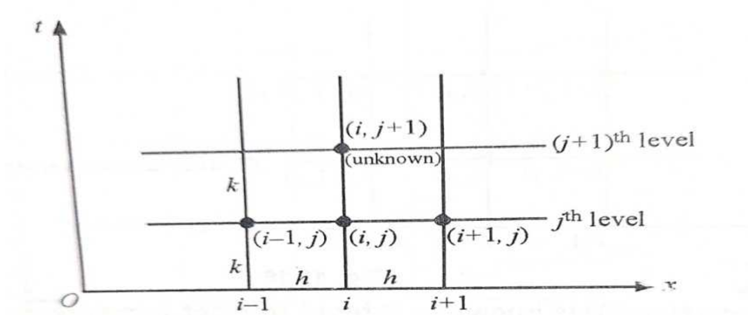

Fig.7

Hence the equation is known as Schmidt Explicit Formula.This formula is valid for $\alpha > 0$ and $1 - 2\alpha \geq 0$ $i.e.,$ $0 \leq \alpha \leq \frac{1}{2}$ when $\alpha = \frac{1}{2}$, equation (2) can be written as

$$u_{i,j+1} = \frac{1}{2}u_{i-1,j} + \frac{1}{2}u_{i+1,j}$$

$$= \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

Then this equation is called **Bendre-Schmidt Formula**.

12.10 *CRANK-NICOLSON METHOD:-*

The Crank-Nicolson implicit method is a specific numerical technique used for solving partial differential equations (PDEs), particularly those that describe parabolic phenomena. It is an extension of the Crank-Nicolson method, which itself is a combination of implicit and explicit methods. The implicit aspect of the Crank-Nicolson method makes it unconditionally stable and well-suited for solving PDEs.

The above Schmidt explicit method is computational simple and it has limitation. This method is stable is $0 < \alpha \leq \frac{1}{2}$ $i.e.,0 < \frac{kc}{h^2} \leq \frac{1}{2}$ or $k < \frac{1}{2}\frac{h^2}{c^2}$ and h must be kept small in order to obtain reasonable accuracy.

In this method the obtained partial differential equation replaced by the mean of itd finite-difference representations on the $(j+1)th$ and jth time rows. From (2) given that

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{c^2}{2}\left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right]$$

Or

$$-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j+1} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j} \ldots (3)$$

where $\alpha = \frac{kc^2}{h^2}$.

Now equation (3) contains three unknowns and the right side contains three known provital values of u, which is shown in figure 8 given below

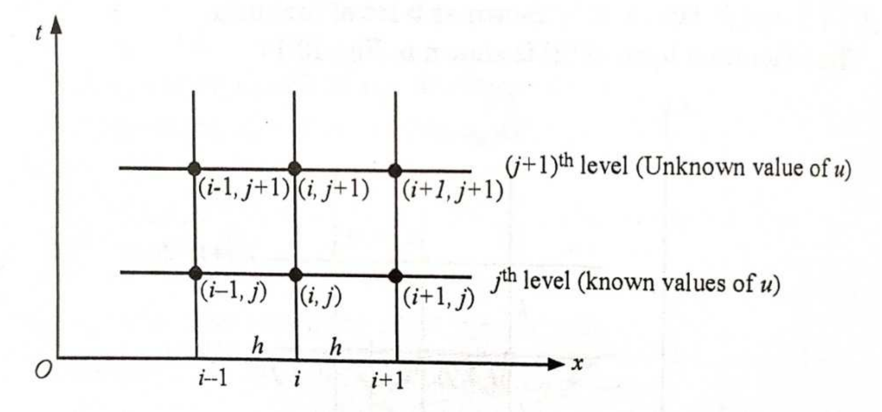

Fig.8

So the equation (3) is called **2-level implicit formula** and is also called **Crank-Nicolson implicit formula.**

For $j = 0$ and $i = 1,23 \ldots \ldots . n$, equation (1.19) generates n simultaneous equations for n unknown pivotal values along the first time row in term of known initial values and boundary values. Thus, for this problem initial and boundary conditions are required.

Similarly, for j = 1, presents n unknown values of u along the second time row in term of calculated values along the first etc. Then the solution of set of simultaneous equations is called as an implicit scheme.

12.11 *DU FORT AND FRANKEL METHOD:-*

If we change $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ in (2) by central difference approximations

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

$$\frac{\partial^2 u}{\partial x^2} = c^2 \left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right]$$

We obtain

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = c^2 \left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right]$$

Or

$$u_{i,j+1} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right] \quad \dots (4)$$

where $\alpha = \frac{kc^2}{h^2}$. This is known as **3-level time formula** and is also known as **Richardson Scheme**.If we replaces $u_{i,j}$ by the average of $u_{i,j-1}$ and $u_{i,j+1}$ $i.e.,\ u_{i,j} = \frac{1}{2}\left[u_{i,j+1} + u_{i,j-1}\right]$ putting in (4), we get

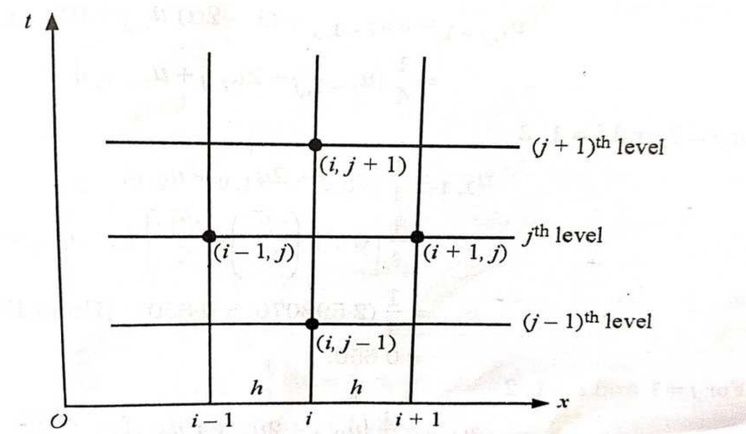

Fig.9

$$u_{i,j+1} = u_{i,} + 2\alpha\left[u_{i-1,j} + (u_{i,j+1} + u_{i,j-1}) + u_{i+1,j}\right]$$

Or

$$u_{i,j+1} = \left(\frac{1-2\alpha}{1+2\alpha}\right)u_{i,j-1} + \left(\frac{2\alpha}{1+2\alpha}\right)\left[u_{i-1,j} + u_{i+1,j}\right] \quad \dots (5)$$

The equation (5) is called **3-level formula** is known as **Du Fort-Frankel**.

EXAMPLE: Solve the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to conditions

$$u(x,0) = sin\pi x, \qquad 0 \le x \le 1$$
$$u(0,t) = u(1,t) = 0$$

using

(i) Schmidt method

(ii) Crank-Nicolson Method
(iii) Du Fort-Franke Method

Carry out computations for the two levels, taking $h = \frac{1}{3},\ k = \frac{1}{36}$

SOLUTION: Here $c^2 = 1, h = \frac{1}{3}, k = \frac{1}{36}$

$$\alpha = \frac{kc^2}{h^2} = \frac{1\left(\frac{1}{36}\right)}{(1/3)^2} = \frac{1}{4}$$

and

$$u_{1,0} = \sin(\pi h) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$u_{2,0} = \sin(2\pi h) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

And another boundary u are zero as shown in figure 10.

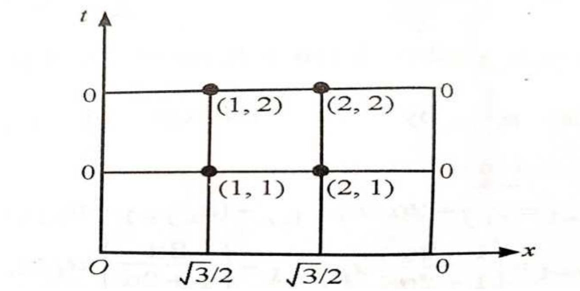

Fig.10

(i) Schmidt method:

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right] \quad \dots(1)$$

For $j = 0,1$ and $i = 1,2$

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right]$$

$$u_{1,1} = \frac{1}{4}\left[0 + 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = 0.650$$

$$u_{1,2} = \frac{1}{4}[0 + 2(0.65) + 0.65] = 0.488$$

$$u_{2,2} = \frac{1}{4}[(0.488) + 2(0.65) + 0] = 0.447$$

(ii) Crank-Nicolson Method:

$$-\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i+1,j+1}$$
$$= \frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i+1,j} \quad \dots(2)$$

Putting the value $j = 0,1$ and $i = 1,2$ in (2) ,we obtain

$$10u_{1,1} - u_{2,1} = \frac{7}{2}\sqrt{3} \quad \ldots (3)$$

$$-u_{1,1} + 10u_{2,1} = \frac{7}{2}\sqrt{3} \quad \ldots (4)$$

$$10u_{1,2} - u_{2,2} = 4.718 \quad \ldots (5)$$

$$-u_{1,2} + 10u_{2,2} = 4.718 \quad \ldots (6)$$

Solving (3) and (4), (5) and (6), we have

$$u_{1,1} = \frac{\frac{77}{2}\sqrt{3}}{99} = 0.674$$

$$u_{2,1} = \frac{\frac{77}{2}\sqrt{3}}{99} = 0.674$$

$$u_{1,2} = \frac{11(4.718)}{99} = 0.524$$

$$u_{2,2} = \frac{11(4.718)}{99} = 0.524$$

(i) Du Fort-Franke Method:

$$u_{i,j+1} = \left(\frac{1-2\alpha}{1+2\alpha}\right)u_{i,j-1} + \left(\frac{2\alpha}{1+2\alpha}\right)\left[u_{i-1,j} + u_{i+1,j}\right]$$

$$= \frac{1}{3}\left[u_{i,j-1} + u_{i-1,j} + u_{i+1,j}\right]$$

For $j = 1$ and $i = 1,2$, we get

$$u_{1,2} = \frac{1}{3}\left[u_{1,0} + u_{0,1} + u_{2,1}\right]$$

$$u_{1,2} = \frac{1}{3}\left[\frac{\sqrt{3}}{2} + 0 + 0.65\right] = 0.505$$

and

$$u_{1,2} = \frac{1}{3}\left[u_{2,0} + u_{1,1} + u_{3,1}\right]$$

$$u_{1,2} = \frac{1}{3}\left[\frac{\sqrt{3}}{2} + 0.65 + 0\right] = 0.505$$

12.12 *HYPERBOLIC EQUATION:-*

A hyperbolic equation is a type of partial differential equation (PDE) that describes wave-like phenomena and is characterized by its hyperbolic partial differential operators. The general form of a hyperbolic equation is often written as:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \dots (1)$$

The hyperbolic nature of these equations arises from the fact that solutions typically involve the propagation of information at finite speeds, similar to wave propagation. Examples of hyperbolic equations include the one-dimensional wave equation and the telegraph equation. These equations are fundamental in describing phenomena such as acoustic waves, electromagnetic waves, and certain types of fluid flow.

12.13 *SOLUTION OF WAVE EQUATION:-*

Let us consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \dots (1)$$

Subject to the initial conditions:

$$u(x,0) = f(x), u_t(x,0) = g(x) \; for \; 0 \le x \le 1$$

And the boundary conditions:

$$u(0,t) = \phi(t), u_t(1,t) = \psi(t) \; for \; 0 \le t \le \infty$$

Again consider a rectangular mesh in $xt - plane$ spacing h along $x - ax\,is$ and the spacing k along $t - axis$. Also $(x,t) = (ih, jk)$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{h^2}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})}{k^2}$$

Substituting these value in (10) , we get

$$\frac{(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})}{k^2} = \frac{c^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

Or

$$u_{i,j+1} = 2(1-\alpha^2)u_{i,j} + \alpha^2(u_{i-1,j} + u_{i+1,j}) - \; u_{i,j-1}$$

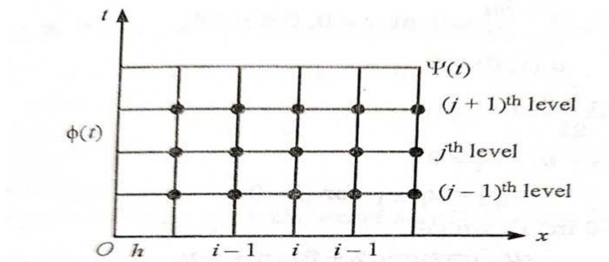

Fig.9

Since we obtain

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} = g(x)$$

$$u_{i,j+1} = u_{i,j-1} + 2kg(x) \ at \ t = 0$$

$$u_{i,1} = u_{i,-1} + 2kg(x) \ at \ t = 0, \ \ for \ j = 0 \qquad \ldots (2)$$

Also $u(x, 0) = f(x)$, this becomes

$$u_{i,-1} = f(x) \qquad \ldots (3)$$

From (2) and (3), we get

$$u_{i,1} = f(x) + 2kg(x)$$

Also the boundary condition become

$$u_{0,j} = \phi(t) \ \text{ and } u_{i,j} = \psi(t)$$

The equation (2) is called 3-level time formula which is convergent when $\alpha < 1 \ i.e., kc < h$.

Remarks:

- If $\alpha = 1 \ i.e., ck = h$, then equation (2) becomes to
$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
- For $\alpha = 1$, the value of u given from (3) is stable and coincides with the solution of (1).
- For $\alpha < 1$, the solution is stable but inaccurate.
- For $\alpha < 1$, the solution is unstable.
- For $\alpha \le 1$, the formula (2) is converges.

EXAMPLE: The transverse displacement $u(x, t)$ of a point at distance x from one end and at any time t of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, with boundary conditions:

$$u(0, t) = 0 = u(4, t)$$

And initial conditions:

$$u(x, 0) = x(4 - x), 0 \le x \le 4$$

$$\frac{\partial u}{\partial t} = 0 \ at \ t = 0 \ and \ 0 \le x \le 4$$

Solve this equation numerically for one half period of vibration, taking $h = 1$ and $= \frac{1}{2}$.

SOLUTION: here $c^2 = 4, h = 1, k = \frac{1}{2}$

$\therefore \qquad \alpha = \frac{ck}{h} = \frac{2\left(\frac{1}{2}\right)}{1} = 1$

Therefore, 3-level time the formula (2) becomes

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \text{... (1)}$$

$$u(0,t) = 0 = u(4,t)$$

$\therefore \qquad u_{0,j} = 0 = u_{4,j} = 0\ \forall j$

Also $u(x,0) = x(4-x)$

$$u_{i,0} = i(4-i) for\ i = 1,2,3$$

So we obtain

$$u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3$$

Also $$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0, 0 \le x \le 4$$

$$u_t(x,0) = 0$$

$\therefore \qquad \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$

$$u_{i,j+1} - u_{i,j-1} = 0$$

$$u_{i,1} = u_{i,-1}\ \ for\ j = 0$$

Now substituting $j = 0$ in (1), we have

$$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right) \qquad \text{... (2)}$$

Substituting $i = 1,2,3$ in (2), we obtain

$$u_{1,1} = \frac{1}{2}(0+4) = 2$$

$$u_{2,1} = \frac{1}{2}(3+3) = 3$$

$$u_{3,1} = \frac{1}{2}(4+0) = 2$$

Again put $j = 1$ in (1) , we given

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3$$

$$u_{1,2} = 2 + 2 - 4 = 0$$

$$u_{3,2} = 3 + 0 - 3 = 0$$

Since the length of vibrating string $= 4 (= l)$

$\therefore \qquad$ Its period $= \frac{2l}{c} = \frac{2(4)}{2} = 4$ so half of its period $= 2$.

<u>SELF CHECK QUESTIONS</u>

1. What is the main difference between partial differential equations (PDEs) and ordinary differential equations (ODEs)?

2. Name three common numerical methods for solving PDEs.
3. How does the finite difference method approximate derivatives in the context of PDEs?
4. Define numerical stability in the context of solving PDEs numerically.
5. Provide an example of a first-order partial differential equation.
6. Give an example of a second-order partial differential equation
7. Differentiate between elliptic, hyperbolic, and parabolic PDEs
8. Why are boundary and initial conditions important in classifying PDEs?
9. What is the main difference between implicit and explicit numerical methods in the context of solving PDEs?
10. Name one advantage of using implicit methods for solving PDEs.
11. How is time stepping handled in implicit methods for parabolic PDEs?
12. Briefly explain the solution process for implicit methods in solving parabolic PDEs.
13. How would you apply an implicit method to solve the heat equation for a material with variable thermal conductivity over time?

12.14 *SUMMARY:-*

In this unit we have studied the finite-difference approximations to derivatives, elliptic equation, method of first approximation value of Laplace equation, solution of Poisson's equation, parabolic equation, an Schmidt explicit method, crank-Nicolson method, du fort and frankel method, hyperbolic equation and solution of wave equation. solving a partial differential equation involves formulating the equation, identifying its type, incorporating boundary or initial conditions, applying appropriate solution techniques such as separation of variables or transforms, solving resulting ordinary differential equations or algebraic equations, verifying the solution, and interpreting the results in the context of the given problem. The solution of a partial differential equation (PDE) refers to the mathematical expression or set of functions that satisfies the given PDE and its associated boundary or initial conditions.

12.15 *GLOSSARY:-*

- **Partial Differential Equation (PDE):** An equation that involves partial derivatives of an unknown function with respect to two or more independent variables.
- **Elliptic PDE:** A type of PDE where the solution represents a steady-state situation, often associated with Laplace's equation.
- **Parabolic PDE:** A type of PDE describing phenomena that evolve over time, such as the heat equation.
- **Hyperbolic PDE:** A type of PDE associated with wave-like behavior and phenomena where information travels at a finite speed, as seen in the wave equation.
- **Boundary Conditions:** Specifications on the behavior of the solution at the boundaries of the spatial domain.
- **Initial Conditions:** Specifications on the behavior of the solution at the initial time for problems involving time evolution.
- **Separation of Variables:** A technique used to simplify PDEs by assuming the solution is a product of functions, each dependent on only one variable.
- **Fourier Transform:** A mathematical transform used to analyze and solve PDEs, particularly for problems involving spatial variables.
- **Laplace Transform:** A mathematical transform often applied to PDEs involving time, converting the problem into an algebraic equation.
- **Green's Function:** A mathematical function used to solve inhomogeneous linear PDEs, providing a way to represent the influence of a localized source.
- **Fundamental Solution:** The solution to a PDE with a point source or impulse, often used as a building block for constructing solutions to more complex problems.
- **Characteristic Curves:** Curves along which the solution of a hyperbolic PDE can be specified, helping to understand the behavior of solutions.
- **Numerical Methods:** Techniques such as finite difference, finite element, or spectral methods used to approximate solutions to PDEs on a computer.

- **Stability Analysis:** Examination of the behavior of a numerical method to ensure that small errors do not grow uncontrollably.
- **Consistency and Convergence:** Properties of numerical methods ensuring that as the discretization becomes finer, the numerical solution approaches the true solution.
- **Verification and Validation:** The process of confirming that a numerical solution accurately represents the physical or mathematical problem it is intended to model.
- **Sensitivity Analysis:** Evaluation of how changes in input parameters or conditions affect the solution of a PDE.

12.16 *REFERENCES:-*

- Randall J. LeVeque(2007),Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems.
- Christian Constanda(2013), Partial Differential Equations: Analytical Solution Techniques
- J. David Logan(2014), Partial Differential Equations: Methods and Applications.
- Richard Haberman(2004), Applied Partial Differential Equations.

12.17 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- https://www.lkouniv.ac.in/site/writereaddata/siteContent/202004032250572068siddharth_bhatt_engg_Numerical_Solution_of_Partial_Differential_Equations.pdf
- S.S.Shastry (2012), Introductory Methods of Numerical Analysis.
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- https://epgp.inflibnet.ac.in/Home/ViewSubject?catid=ZLCHeZEhCZ8yCri36nSF3A==

12.18 *TERMINAL QUESTIONS:-*

(TQ-1): Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the initial condition $u = sin\pi x$ at $t = 0$ for $0 \leq x \leq 1$ and $u = 0$ at $x = 0$ and $x = 1$ for $t > 0$, by the Gauss-Seidel method.

(TQ-2): Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the domain of the following figure by

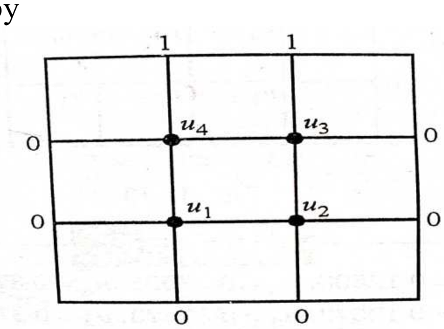

Fig.10

a. Jacobi's method
b. Gauss-Seidel method.

(TQ-3): Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the square mesh with boundary value shown in the following figure:

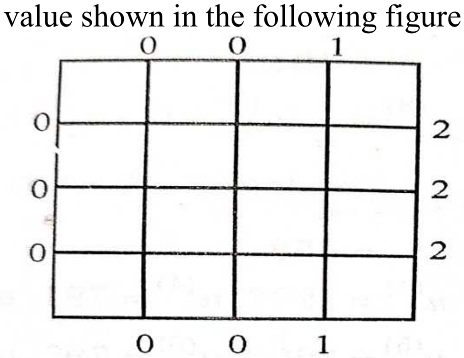

Fig.11

(TQ-4): Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the domain of figure:

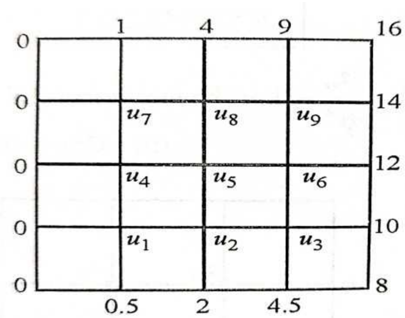

Fig.12

(TQ-5): Find the solution of parabolic equations

$$2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

Gives that $u(0,t)=u(4,t)=0$ and $u(x,0)=4x-x^2=0$ taking $h=1$. Find the values of u upto $t=5$.

(TQ-6): Solve the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ subject to conditions $u(0,t)=u(1,t)=0$ and

$$u(x,0)=\begin{cases} 2x, & 0\le x\le \frac{1}{2} \\ 2(1-x), & \frac{1}{2}\le x\le 1 \end{cases}$$

(TQ-7): Evaluate the pivotal values of equation $\frac{\partial^2 u}{\partial t^2}=16\frac{\partial^2 u}{\partial x^2}$ taking $h=1$ up to $t=1.25$. The boundary conditions are

$$u(0,t)=u(5,t)=0, u_t(x,0)=0$$

and $$u(x,0)=x^2(5-x)$$

(TQ-8): The function satisfied the equation $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ and the conditions:

$$u(x,0)=\frac{1}{8}sin\pi x, u_t(x,0)=0\ for\ 0\le x\le 1$$

$$u(0,t)=u(1,t)=0\ for\ t\ge 0$$

Use the explicit scheme to calculate u for $x=0$ to 1 and $t=0$ to 0.5 with $h=0.1, k=0.1$

12.19 *ANSWERS:-*

SELF CHECK ANSWERS

1. PDEs involve functions of multiple independent variables, while ODEs involve functions of a single independent variable.
2. Finite Difference Method, Finite Element Method, and Finite Volume Method.

3. It approximates derivatives by discretizing the domain into a grid and replacing derivatives with finite difference approximations.
4. Numerical stability refers to the ability of a numerical method to produce accurate results without amplifying errors over time.
5. Example: $u_x + u_y = 0$, (Transport Equation).
6. Example: $u_{xx} + u_{yy} = 0$ (Laplace's Equation) or $u_{tt} - c^2 u_{xx} = 0$ (Wave Equation).
7. Elliptic PDEs have a steady-state solution, hyperbolic PDEs describe wave-like behavior, and parabolic PDEs involve diffusion over time.
8. Boundary and initial conditions specify the behavior of the solution at the boundaries and at the initial time, helping in uniquely determining the solution.
9. In implicit methods, future values depend on both current and future states, while in explicit methods, they depend only on current states.
10. Implicit methods are unconditionally stable, allowing for larger time steps without the constraint imposed by stability conditions in explicit methods.
11. Implicit methods use backward differencing in time, resulting in a system of equations that must be solved at each time step.
12. Implicit methods involve solving a system of linear equations, often done using iterative numerical techniques such as the Crank-Nicolson method.
13. The implicit method would involve discretizing both the spatial and temporal derivatives, resulting in a system of equations to solve for the temperature distribution.

TERMINAL ANSWERS

(TQ-2):a. $u_1 = 0.12598,\ u_2 = 0.12598,, u_3 = 0.37598,$
$u_4 = 0.37598$

b. $u_1 = 0.12646,\ u_2 = 0.12573,, u_3 = 0.37573,$
$u_4 = 0.37573$

(TQ-3): $u_1 = 26.66,\ u_2 = 33.33,, u_3 = 43.33,\ u_4 = 46.66$

(TQ-4): $u_1 = 1.57,\ u_2 = 3.71,, u_3 = 6.57, u_4 = 2.06, u_5 = 4.69,$
$u_6 = 8.06,\ u_7 = 2.00, u_8 = 4.96, u_9 = 9.00$

(TQ-5):

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1·5	2	1·5	0
3	0	1	1·5	1	0
4	0	0·75	1	0·75	0
5	0	0·5	0·75	0·5	0

(TQ-6):

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	0·09	0·16	0·21	0·24	0·25	0·24	0·21	0·16	0·09	0
1	0	0·08	0·15	0·20	0·23	0·24	0·23	0·20	0·15	0·08	0
2	0	0·075	0·14	0·19	0·22	0·23	0·22	0·19	0·14	0·75	0
3	0	0·07	0·13	0·18	0·21	0·22	0·21	0·18	0·13	0·07	0

(TQ-7):

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	– 2	0
3	0	2	– 2	– 8	– 7	0
4	0	– 9	– 14	– 11	– 6	0
5	0	– 16	– 18	– 12	– 4	0

(TQ-8):

t \ x	0	0·1	0·2	0·3	0·4	0·5
0·1	0	0·037	0·07	0·096	0·113	0·119
0·2	0	0·031	0·059	0·082	0·096	0·101
0·3	0	0·023	0·043	0·059	0·07	0·074
0·4	0	0·012	0·023	0·031	0·037	0·039
0·5	0	0	0	0	0	0

Unit 13: Applications to Integral Equations

CONTENTS:

13.1 *INTRODUCTION:-*

The theory of Integral Equations holds a significant position within the realm of Mathematics, serving as a powerful tool for addressing a wide range of initial and boundary value problems associated with both ordinary and partial differential equations. This mathematical framework allows for the reduction of complex problems to integral equations of diverse types. The evolution of integral equations is deeply intertwined with the historical development of mathematics, particularly in the field of applied mathematics.

The origins of integral equations can be traced back to N.H. Abel in 1826, who pioneered the reduction of a physical problem involving the descent of a particle along a smooth vertical curve under the influence of gravity within a specific time interval. Abel's groundbreaking work paved the way for further advancements, including V. Volterra's contributions in 1896. Volterra extended the theory by developing a general solution for a class of linear integral equations, particularly those with a variable upper limit of integration, now commonly referred to as Volterra integral equations. Building upon these foundations, I. Fredholm in 1900 made significant strides in the theory of integral equations. His focus was on integral equations with constant limits of integration, and these equations are now recognized as Fredholm integral equations. This historical progression

highlights the collaborative efforts of notable mathematicians in shaping the theory of integral equations, showcasing its evolution and application within the broader context of mathematical and applied sciences.
In this unit we will studied about the integral equations and their application.

13.2 *OBJECTIVES:-*

The objectives of applications to integral equations can vary depending on the specific context and problem being addressed. The integral equations encompass a broad range of goals, including modeling physical phenomena, finding analytical and numerical solutions, studying linear operators, addressing optimization and inverse problems, analyzing singularities, handling boundary value problems, and contributing to advancements in mathematical methods. These applications have far-reaching implications across various scientific and engineering disciplines. After studying this unit, you will be able to

- Formulate physical, engineering, or scientific problems as integral equations.
- Address boundary value problems through integral equations.
- Apply integral transforms to simplify integral equations.

The applications of integral equations are broad and impactful, providing powerful tools for understanding, modeling, and solving complex problems across various scientific and engineering disciplines.

13.3 *INTEGRAL EQUATION:-*

An integral equation is one in which function to be established appears under the integral sign. The most general form of a linear integral equation is

$$F(t) = G(t) + \lambda \int_a^b K(t,u)F(u)du$$

where the upper limit may be either variable or fixed. The function $G(t)$ and $K(t,u)$ are called functions, while $F(t)$ is to be determined. The function is called $K(t,u)$ is called the kernel of the integral equation.

If a and b are constants, the equation is called the *Fredholm integral equation*. If a is a constant while b=t, it is called a *Volterra integral equation.*

13.4 *LINEAR AND NONO-LINEAR INTEGRAL EQUATIONS:-*

A linear integral equation is an equation in which the unknown function and its derivatives appear linearly (to the power of one) within the integral expression. The equation can be expressed as a linear combination of the unknown function and its integrals. If integral equation is not linear then it is called non-linear integral equation.

For Example:

$$f(x) = \int_a^x K(x,t)\, g(t)dt \qquad \dots (1)$$

$$g(x) = f(x) + \int_a^b K(x,t)\, g(t)dt \quad \dots (2)$$

$$g(x) = \int_a^b K(x,t)\, \big(g(t)\big)^2 dt \qquad \dots (3)$$

where $a \le x \le b, \quad a \le t \le b.$

The above integral equation (1), (2) are linear while the equation (3) is non-linear.
The general form of a linear integral equation is

$$\alpha(x)g(x) = f(x) + \lambda \int_\Omega K(x,t)\, g(t)dt \quad \dots (4)$$

where the function f, α and K are known functions, while g is to be determined, λ is non-zero real and complex parameter. The function $K(x,t)$ is known as kernel of the integral equation and the domain Ω of the auxiliary variable t.

An integral equation, which linear involve the linear operator

$$L[\quad] = \int_\Omega K(x,t)[\quad]dt$$

Having the kernel $K(x,t)$. It satisfied the linearity condition

$$L\{c_1 g_1(t) + c_2 g_2(t)\} = c_1 L\{g_1(t)\} + c_1 L\,\{g_2(t)\}$$

where $L\{g(t)\} = \int_\Omega K(x,t)g(t)dt$ and c_1, c_2 are constants.

13.5 *SOME SEPECIAL TYPES OF INTEGRAL EQUATIONS:-*

i. **<u>Volterra Integral equation of first and second kinds:</u>** A linear integral equation is called **Volterra Integral equation** if the upper limit of integration is a variable. e.g.

$$\alpha(x)g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)dt \quad \text{... (5)}$$

where a is a constant, $f(x), \alpha(x)$ and $K(x,t)$ are known functions while $g(x)$ is unknown function, λ is a non-zero real or complex parameter and the equation (5) is called **Volterra Integral equation.**

a. If $\alpha = 0$, the unknown function g appears only under the integral sign , then from (5), we have

$$f(x) + \lambda \int_a^x K(x,t)\, g(t)dt = 0 \quad \text{... (6)}$$

is known as **Volterra Integral equation** of first kind.

b. If $\alpha = 0$, the equation (5) say, we get

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)dt \quad \text{... (7)}$$

is known as **Volterra Integral equation** of Second kind.

c. If $\alpha = 1, f(x) = 0$, the equation (5) reduces to

$$g(x) = \lambda \int_a^x K(x,t)\, g(t)dt$$

is called the homogeneous **Volterra Integral equation** of Second kind.

ii. **<u>Fredholm Integral equation of first, second and third kinds:</u>** A linear integral equation is called **Fredholm Integral equation** if the upper limit of integration is fixed. e.g.

$$\alpha(x)g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)dt \quad \text{... (8)}$$

where a and b is a constants, $f(x), \alpha(x)$ and $K(x,t)$ are known functions while $g(x)$ is unknown function, λ is a non-zero real or complex parameter and the equation (5) is called **Fredholm Integral equation of third kind.**

a. If $\alpha = 0$, the equation (8) unknown function g only under the integral sign , then, we obtain

$f(x) + \lambda \int_a^b K(x,t)\, g(t)dt = 0$ ... (9)

is known as **Fredholm Integral equation of first kind.**

b. If $\alpha = 1$, the equation (9) say, we get

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)dt \quad ... (10)$$

is known as **Fredholm Integral equation of first kind.**

c. If $\alpha = 1, f(x) = 0$, the equation (8) reduces to

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)dt$$

is called the **Homogeneous Integral Equation of Second kind.**

iii. **Singular Integral equation:** A singular integral equation is defined as an equation in which one or both of the integration limits extend to infinity, or the kernel (the function defining the integral) becomes infinite at one or more points within the integral under consideration.

For Example:

$$f(x) = \int_a^x sin(x,t)\, g(t)dt$$

$$g(x) = f(x) + \int_{-\infty}^{\infty} K(x,t)\, g(t)dt$$

$$f(x) = \int_a^x \frac{K(x,t)}{(x-t)^\alpha} g(t)dt, \qquad 0 < r < 1$$

$$f(x) = \int_a^x \frac{g(t)}{(x-t)^\alpha} dt, \qquad 0 < \alpha < 1$$

are singular integral equations.

iv. **Integral equation of convolution type:** An integral equation

$$g(x) = f(x) + \lambda \int_0^t K(x-t)g(t)dt$$

in which the kernel $K(t-x)$ is a function of the difference $t - x$ only, and corresponding Fredholm integral equation

$$g(x) = f(x) + \lambda \int_a^b K(x-t)g(t)dt$$

are called integral equation of the convolution type.

13.6 *SPECIAL OR DEGENERATE KERNEL:-*

i. **Separable or Degenerate Kernel:** A kernel $K(x,t)$ is considered separable or degenerate if it can be expressed as the sum of a finite number of terms, and each term is a product of a function that depends solely on x and a function that depends solely on t. In mathematical terms, a separable or degenerate kernel is represented as:

$$K(x,t) = \sum_{i=1}^{n} a_i\,(x) b_i(t)$$

Here, $a_i(x)$ represents a function of x only, $b_i(t)$ represents a function of t represents a function of x only and the sum is taken over a finite number of term (n).

ii. **Symmetric Kernel:** A complex-valued function $K(x,t)$ is termed symmetric or Hermitian if it satisfies the condition$K(x,t) = \overline{K(x,t)}$, where the bar denotes the complex conjugate. In the case of a real kernel, this symmetry condition simplifies to $K(x,t) = \overline{K(x,t)}$, aligning with the conventional definition of symmetry.

13.7 *SOME IMPORTANT RESULTS:-*

i. **Convolution(or Falting):** The convolution of $F(t)$ and $G(t)$ is denoted and described as

$$F * G = \int_0^t F(x)G(t-x)dx$$

Or

$$F * G = \int_0^t F(t-x)G(x)dx$$

ii. **Convolution theorem or convolution property:** If $L\{F(t) = f(s)\}$ and $L\{G(t) = g(s)\}$, then

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(t-x)G(x)dx = F * G$$

So

$$L(F * G) = f(s)g(s) = L[F(t)] \times L[G(t)]$$

$i.e.,$

$$L\left\{\int_0^t F(x)G(t-x)dx\right\} = L\left\{\int_0^t F(t-x)G(x)dx\right\} = f(s)g(s)$$

iii. The Abel Integral Equation: An integral equation is the form

$$\int_0^t \frac{F(u)}{(t-u)^{\alpha}} du = G(t)$$

is called Abel's integral equation, where $F(t)$ is unknown function, $G(t)$ is known function and α is constant i.e., $0 < \alpha < 1$.

iv. Integro-differential equation: An integro-differential equation is a type of mathematical equation that involves both differential operators and integral operators. These equations combine elements of differential equations and integral equations, making them more complex and challenging to solve than either type alone.

An equation in which various derivatives of known function $F(t)$ can also be written as

$$F''(t) = F(t) + G(t) + \int_0^t K(t-u)F(u)du$$

is an integro-differential equation, where $F(t)$ is unknown function, $G(t)$ and $K(t-u)$ is known function. This solution of such equation subject to given initial conditions can be easily written.

13.8 *APPLICATION OF LAPLACE TRANSFORM IN SOLVING VOLTERRA INTEGRAL EQUATION:-*

Application of Laplace Transform in solving Volterra integral equation with convolution of kernel and the working rule is given by

i. Suppose the volterra integral equation of first kind of the form is given as given below

$$G(t) = \lambda \int_a^t K(t,u)F(u)du$$

Or

$$G(t) = \lambda K(t) * F(t) \qquad \ldots (1)$$

Suppose

$$L\{G(t)\} = g(s), \quad L\{F(t)\} = f(s) \quad \text{and} \quad L\{K(t)\} = k(s) \quad \ldots (2)$$

From (1), we get

$$L\{G(t)\} = \lambda L\{K(t) * F(t)\}$$

Or

$$L\{G(t)\} = \lambda\, L\{K(t)\} \times L\{F(t)\}$$

Since from (2), we obtain

$$g(s) = \lambda k(s) f(s) \qquad or \qquad f(s) = \frac{g(s)}{\lambda k(s)} \qquad \ldots (3)$$

Now according the inverse Laplace transform of both sides of (3), we have

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\} \qquad or \qquad F(t) = L^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\}$$

ii. Suppose the volterra integral equation of second kind of the form is obtained below

$$F(t) = G(t) + \lambda \int_0^t K(t-u) F(u) du$$

$$F(t) = G(t) + \lambda K(t) * F(t) \qquad \ldots (1)$$

Now suppose $L\{G(t)\} = g(s), L\{F(t)\} = f(s)$ and

$$L\{K(t)\} = k(s) \qquad \ldots (2)$$

Putting the Laplace transform of both sides of (1), we obtain

$$L\{F(t)\} = L\{G(t)\} + \lambda L\{K(t) * F(t)\}$$

$$L\{F(t)\} = L\{G(t)\} + \lambda L\{K(t)\} \times L\{F(t)\}$$

So

$$f(s) = g(s) + \lambda k(s) f(s) \qquad or \qquad [1 - \lambda k(s)] f(s) = g(s)$$

Hence

$$f(s) = g(s)/[1 - \lambda k(s)]$$

Again the inverse transform of both sides of above equation is

$$F(t) = L^{-1}\{g(s)/[1 - \lambda k(s)]\}$$

SOLVED EXAMPLE

EXAMPLE1: Solve the integral equation $F(t) = 1 + \int_0^t f(u) sin(t - u)\, du$ and verify your solution.

SOLUTION: Given

$$F(t) = 1 + \int_0^t f(u) sin(t-u)\, du \qquad ...(1)$$

From (1) can be written as

$$F(t) = 1 + F(t) * sint \qquad ...(2)$$

Let $L\{F(t) = f(s)\}$ obtaining the Laplace transformation of (2), we have

$$L\{F(t)\} = L\{1\} + L\{F(t) * sint\}$$

$$L\{F(t)\} = \left(\frac{1}{s}\right) + L\{F(t)\} \times L\{sint\}, \qquad by\ convolution\ theorem$$

$f(s) = \frac{1}{s} + f(s) \times \frac{1}{s^2+1}$ so $\left(1 - \frac{1}{s^2+1}\right) f(s) = \frac{1}{s}$\

$$f(s) = \frac{(s^2+1)}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

Applying the inverse transform of both sides of (3), we obtain

$$\left.\begin{aligned} L^{-1}\{f(s)\} &= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\} \\ F(t) &= 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2} \end{aligned}\right\} \qquad ...(2)$$

Verification of (2): The equation (2) satisfies the given integral equation (1), we have

$$F(u) = 1 + \frac{u^2}{2}$$

Take R.H.S. of (1), we obtain

$$= 1 + \int_0^t \left(1 + \frac{u^2}{2}\right) \sin(t-u)\, du$$

$$= 1 + \left[\left(1 + \frac{u^2}{2}\right)\cos(t-u)\right]_0^t - \int_0^t u \cos(t-u)\, du$$

$$= 1 + 1 + \frac{t^2}{2} - cost - \left\{[-usin(t-u)]_0^t - \int_0^t 1.(-sin(t-u))\, du\right\}$$

$$= 2 + \frac{t^2}{2} - cost - \int_0^t sin(t-u)\, du$$

$$= 2 + \frac{t^2}{2} - cost - [\cos(t-u)]_0^t = 2 + \frac{t^2}{2} - cost - (1 - cost)$$

$$= 2 + \frac{t^2}{2} = F(t), \qquad using\ (2)$$

$$= L.H.S. of\ (2)$$

Hence the equation (2) is the solution of obtained integral equation (1).

EXAMPLE2: Solve by the method of Laplace transform: $3sin2x = y(x) + \int_0^x (x-t)\, y(t)\, dt$ and verify your solution.

SOLUTION: Take special note of the fact that x is used in place of t and t is used in place of u in the current problem. Therefore, the standard solution and outcomes should be adjusted accordingly.

By applying the convolution definition, the following equation can be found:

$$y(x) = 3sin2x - y(x) * x \qquad ...(1)$$

Let $L\{y(x) = f(s)\}$. Applying the Laplace transform of both sides of (1), we obtain

$$L\{y(x)\} = 3L\{sin2x\} - L\{y(x) * x\} \qquad ...(2)$$

$$f(s) = 3 \times \left\{\frac{2}{s^2+2^2}\right\} - L\{y(x)\} \times L\{x\}$$

$$f(s) = \frac{6}{s^2+4} - \frac{f(s)}{s^2}$$

$$f(s) = \frac{6s^2}{(s^2+4)(s^2+1)}$$

$$f(s) = 2\left\{\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right\}$$

Taking inverse Laplace integral transform of both sides of (2), we have

$$y(x) = 2\left\{sint - \frac{sin2t}{2}\right\} = 2sint - sin2t$$

EXAMPLE3: Solve the Abel's integral equation $\int_0^t \frac{F(u)}{\sqrt{(t-u)}} du = 1 + t + t^2$.

SOLUTION: Let the given equation

$$\int_0^t \frac{F(u)}{\sqrt{(t-u)}} du = 1 + t + t^2 \qquad ...(1)$$

$t^{-1/2} * F(t) = 1 + t + t^2$, by the definition of convolution.

Let $L\{F(t)\} = f(s)$. Applying the Laplace transform to both sides of (1), we obtain

$L\left\{t^{-\frac{1}{2}}\right\} \times L\{F(t)\} = L\{1\} + L\{t\} + L\{t^2\}$, using convolution theorem

$$\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} f(s) = \frac{1}{s} + \frac{1!}{s^2} + \frac{2!}{s^3}$$

$f(s) = \frac{s^{\frac{1}{2}}}{\sqrt{\pi}}\left(\frac{1}{s} + \frac{1!}{s^2} + \frac{2!}{s^3}\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{s^{\frac{1}{2}}} + \frac{1}{s^{\frac{3}{2}}} + \frac{1}{s^{\frac{5}{2}}}\right)$ as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Applying the inverse transform of both sides , we obtain

$$F(t) = \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{2t^{3/2}}{\Gamma(5/2)}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{t^{1/2}}{\left(\frac{1}{2}\right)\times\sqrt{\pi}} + \frac{2t^{3/2}}{\left(\frac{3}{2}\right)\times\left(\frac{1}{2}\right)\times\sqrt{\pi}}\right]$$

Thus $\qquad F(t) = \left(\frac{1}{\pi}\right)\times\left[t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} + \left(\frac{8}{3}\right)\times t^{\frac{3}{2}}\right]$

EXAMPLE4: Solve the integro- differential equation

$$F'(t) = sint + \int_0^t F(t-u)cosudu$$

Where $F(0) = 0$.

SOLUTION: Let the given equation

$$F'(t) = sint + \int_0^t F(t-u)cosudu$$

Or

$F'(t) = sint + F(t) * cost,$ by the definition of convolution

Also, obtained that

$$F(0) = 0$$

Let $L\{F(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$$L\{F'(t)\} = L\{sint\} + L\{F(t) * cost\}$$

$$sL\{F(t)\} - F(0) = \frac{1}{(1+s^2)} + L\{F(t)\}\times L\{cost\}$$

$$sf(s) = \frac{1}{(1+s^2)} + f(s) * \left[\frac{s}{(1+s^2)}\right], \text{ using (2)}$$

$$\left(1 + \frac{1}{1+s^2}\right)sf(s) = \frac{1}{1+s^2}$$

$$f(s) = \frac{1}{s^3}$$

Now,

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$F(t) = L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!} = \frac{t^2}{2}.$$

SELF CHECK QUESTIONS

1. Define the following
   a. Singular integral equation
   b. The Abel's integral equation

c. Integro-differential equation
d. Integral equation of convolution type.
e. Fredholim and Volterra integral equation of first and second kinds.

2. State whether the following statements are true or false.
    a. The presence of a singularity in the kernel function always leads to a solution that is not well-defined.
    b. A unique solution to a Fredholm integral equation of the second kind is guaranteed if and only if the constant λ is an eigenvalue of the integral operator.
    c. The kernel function in a double integral Fredholm integral equation of the second kind depends on both the independent variables x and y as well as the unknown function u(x,y).
    d. A Fredholm integral equation of the second kind involves an integral of the unknown function over the same domain as the kernel function.
    e. In a Volterra integral equation, the limits of integration are fixed constants.

13.9 *SUMMARY:-*

Integral equations are mathematical equations that involve unknown functions within integrals. They come in linear and nonlinear forms, with applications in physics, engineering, and other fields. Solving these equations helps model relationships with spatial or temporal dependencies using various numerical and analytical methods.

Integral equations play a crucial role in mathematical modeling, providing a powerful tool for describing relationships between quantities that involve integration. They have applications in diverse fields, and their solutions contribute to our understanding of various natural phenomena.

13.10 *GLOSSARY:-*

- **Integral Equation:** A mathematical equation that involves an unknown function within an integral. It expresses a relationship between a function and the integral of that function.
- **Kernel:** The function that defines the integrand in an integral equation. It represents the interaction between different parts of the unknown function.
- **Fredholm Equation:** An integral equation where the kernel is a given function. Fredholm equations often arise in the study of linear integral equations.

- **Volterra Equation:** An integral equation where the kernel depends on the solution itself. Volterra equations are often encountered in problems with nonlinear dependencies.
- **Linear Integral Equation:** An integral equation in which the unknown function appears linearly. The linearity simplifies the analysis and solution methods.
- **Nonlinear Integral Equation:** An integral equation in which the unknown function appears nonlinearly. Nonlinear integral equations are more complex and may require specialized techniques for solution.
- **Definite Integral Equation:** An integral equation with constant limits of integration. The definite integral defines the range over which the integration occurs.
- **Integro-Differential Equation:** An equation that includes both derivatives and integrals of the unknown function. These equations capture both spatial and temporal dependencies.
- **Method of Moments:** A numerical technique for solving integral equations by matching the moments of the kernel. It involves choosing specific functions to satisfy certain conditions.
- **Collocation Method:** A numerical approach that selects specific points in the domain to evaluate the integral equation. The equation is satisfied at these selected points.
- **Quadrature Methods:** Numerical techniques that approximate integrals using weighted sums at discrete points. These methods are often used for solving integral equations numerically.
- **Analytical Methods:** Techniques involving algebraic and calculus manipulations to find exact solutions to integral equations. These methods provide closed-form expressions for the solutions.
- **Numerical Methods:** Computational techniques for approximating solutions to integral equations, including discretization and iterative methods. These methods are employed when analytical solutions are not feasible.
- **Application Areas:** Physics, engineering, biology, and economics are among the fields where integral equations find applications. They are used to model a wide range of physical and natural phenomena involving integration.

13.11 *REFERENCES:-*

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13.12 *SUGGESTED READING:-*

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13.13 *TERMINAL QUESTIONS:-*

(TQ-1): Give the definition and complete classification of linear integral equation.

(TQ-2): Show that the function $g(x) = 1$ is a solution of the Fredholm integral equation.

$$g(x) = \int_0^1 x(e^{xt} - 1)g(t)dt = e^x - x$$

(TQ-3): Show that the function $g(x) = \frac{1}{2}$ is a solution of the integral equation

$$\int_0^1 \frac{g(t)}{\sqrt{x-t}}dt = \sqrt{x}$$

(TQ-4): Show that the function $g(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation

$$\int_0^1 \frac{g(t)}{\sqrt{x-t}}dt = 1$$

(TQ-5): Show that the homogeneous integral equations

i. $g(x) = \lambda \int_0^1 x\left(t\sqrt{x} - x\sqrt{t}\right)g(t)dt$

ii. $g(x) = \lambda \int_0^1 x(3x-2)t\, g(t)dt$

(TQ-6): Solve the integral equation $F(t) = t + 2\int_0^t F(u)\cos(t-u)du$.

(TQ-7): Solve the integral equation $F(t) = t^2 + 2\int_0^t F(u) sin\,(t-u)du$.

(TQ-8): Solve the following integral equations:

a. $F(t) = e^{-t} - 2\int_0^t F(u)\cos\,(t-u)du$.

b. $e^{-x} = y(x) + 2\int_0^x \cos(x-t)\,y(t)dt$

c. $F(t) = asint - 2\int_0^t F(u)\cos\,(t-u)du$

d. $t = \int_0^t e^{t-u}F(u)du$

(TQ-9): Solve the following Abel's equation

a. $\int_0^t \frac{F(u)}{\sqrt{(t-u)}}du = 1 + t + t^2$

b. $\int_0^t \frac{F(u)}{(t-u)^{1/3}}du = 1(1+t)$

c. $G(t) = \int_0^t \frac{F(u)}{(t-u)^{\alpha}}du\,, 0 < \alpha < 1$

(TQ-10): Solve the integral equation $F'(t) = t + 2\int_0^t F(t-u)cos\,u\,du$, if $F(0) = 4$.

(TQ-11): Show that the function $g(x) = e^x\left(2x - \frac{2}{3}\right)$ is a solution of the Fredholm equation.

(TQ-12): Show that the function $g(x) = (1+x^2)^{-3/2}$ is the solution of Volterra integral equation.

(TQ-13): Show that the function $g(x) = sin\,(\pi x/2)$ is the solution of the Fredholm integral equation.

13.13 *ANSWERS:-*

SELF CHECK ANSWERS

1.

a. Singular Integral Equation: A singular integral equation is a type of integral equation in which the kernel (the function inside the integral) becomes singular at some point or over some interval. Singular integral equations often arise in various branches of mathematics and physics, and their solutions may require specialized techniques to handle the singularities.

For Example:

$$f(x) = \int_a^x sin(x,t)\,g(t)dt$$

$$g(x) = f(x) + \int_{-\infty}^{\infty} K(x,t)\, g(t)dt$$

$$f(x) = \int_a^x \frac{K(x,t)}{(x-t)^\alpha} g(t)dt, \qquad 0 < r < 1$$
$$f(x) = \int_a^x \frac{g(t)}{(x-t)^\alpha} dt, \qquad 0 < \alpha < 1$$

are singular integral equations.

b. Abel's Integral Equation: Abel's integral equation is a specific type of integral equation named after the Norwegian mathematician Niels Henrik Abel. The equation typically involves an unknown function and an integral containing the product of the unknown function and another function. Abel's integral equation arises in various mathematical and physical contexts, and solving it often requires specific methods depending on the given conditions.

An integral equation is the form

$$\int_0^t \frac{F(u)}{(t-u)^\alpha} du = G(t)$$

is called Abel's integral equation, where $F(t)$ is unknown function, $G(t)$ is known function and α is constant i.e., $0 < \alpha < 1$.

c. Integro-Differential Equation: An integro-differential equation is an equation that combines both differential and integral operators. It involves a function and its derivatives along with an integral of the function. Integro-differential equations are used to model a wide range of phenomena in physics, engineering, and applied mathematics. Solving integro-differential equations can be challenging and may require a combination of techniques from differential equations and integral equations.

An equation in which various derivatives of known function $F(t)$ can also be written as

$$F''(t) = F(t) + G(t) + \int_0^t K(t-u)F(u)du$$

is an integro-differential equation, where $F(t)$ is unknown function, $G(t)$ and $K(t-u)$ is known function.

d. Integral Equation of Convolution Type: An integral equation of convolution type involves a convolution operation within the integral. The convolution of two functions is a mathematical operation that expresses the integral of the product of the functions, after one is reversed and shifted. Convolution-type integral equations often appear in signal processing, image processing, and other areas where the interaction between different signals or functions is essential.

An integral equation

$$g(x) = f(x) + \lambda \int_0^t K(x-t)g(t)dt$$

in which the kernel $K(t-x)$ is a function of the difference $t-x$ only, and corresponding Fredholm integral equation

$$g(x) = f(x) + \lambda \int_a^b K(x-t)g(t)dt$$

are called integral equation of the convolution type.

2.

a. F
b. F
c. F
d. T
e. T

TERMINAL ANSWERS

(TQ-6): $F(t) = 2 + t - 2e^t + 2e^t t = 2 + t - 2e^t(1-t)$

(TQ-7): $F(t) = 2 + \frac{t^2}{2!} + 2\frac{t^4}{4!} + 2e^t t = t^2 + \frac{t^4}{12}$

(TQ-8): a. $F(t) = e^{-t}(1-t)^2$

b. $y(x) = e^{-x}(1-x)^2$

c. $F(t) = ae^{-t} \times \frac{t}{1!} = ate^{-t}$

d. $F(t) = 1 - t$

(TQ-1): a. $F(t) = \left(\frac{1}{\pi}\right) \times \left[t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} + \left(\frac{8}{3}\right) \times t^{\frac{3}{2}}\right]$

b. $F(t) = \left(\frac{3\sqrt{3}t^{\frac{1}{3}}(2+3t)}{4\pi}\right)$

c. $F(t) = \frac{sin\pi\alpha}{\pi}\frac{d}{dt}\left\{\int_0^t (t-u)^{\alpha-1}F(u)du\right\}$

(TQ-10): $F(t) = 4 + 5\frac{t^2}{2} + \frac{t^4}{24}$

Unit 14: Finite Element Method

CONTENTS:

14.1 *INTRODUCTION:-*

In Unit 12, we have discussed the finite difference method for finding solutions of partial differential equations. In particular, we have solved the Laplace equation, Poisson equation and one-dimensional heat and wave equations. The introduction provides an overview of the finite element method as a widely used numerical technique for solving partial differential equations (PDEs). It highlights the method's applicability to both initial and boundary value problems in both ordinary and partial differential equations. However, the focus of the discussion in this unit is on boundary value problems, specifically addressing the finite element methods for solving Laplace and Poisson equations in two dimensions. Furthermore, the introduction emphasizes the advantages of the finite element method over the finite difference method, particularly in handling boundary conditions. It notes that the finite element method offers relative ease in managing boundary conditions, especially for irregularly shaped boundaries, compared to the finite difference method, which requires the development of special formulas for boundary treatment.

In this unit outlines the initial topics covered, starting with the definition of triangular and rectangular finite elements commonly employed in two-dimensional problems. It also introduces Galerkin's finite element method, a weighted residual method, and illustrates its application to solving the

Dirichlet's boundary value problem for the Poisson equation and Laplace equation. Additionally, the unit discusses the variational formulation of the Laplace and Poisson equations, providing a comprehensive foundation for further exploration of finite element methods in solving partial differential equations.

14.2 *OBJECTIVES:-*

After studying this unit you should be able to

- To break down a given two-dimensional domain into triangular and rectangular finite elements.
- To develop the finite element Galerkin method specifically tailored for solving Laplace and Poisson equations with Dirichlet boundary conditions.

14.3 *FINITE ELEMENTS:-*

Finite element method (FEM) is a numerical technique used to solve partial differential equations (PDEs) and integral equations. It's widely employed in engineering and physics for simulating physical phenomena like heat transfer, fluid flow, and structural mechanics. The method subdivides a complex system into smaller, simpler parts called finite elements. Equations governing the behavior within each element are formulated, resulting in a system of algebraic equations. These equations are then solved to approximate the behavior of the entire system. FEM is versatile, allowing for the analysis of structures with irregular shapes and complex material properties. In finite element methods, we produce difference equations by using the variational principle or weighted residual methods. The closed domain R, where they obtained partial differential equation holds, is divided into a finite number of non-overlapping subdomains $R_1, R_2 \dots\dots.. R_n$. These subdomains are known as the finite elements.

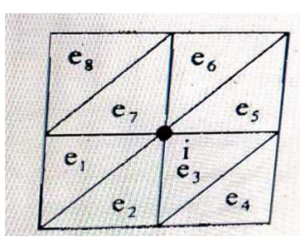

Fig.1

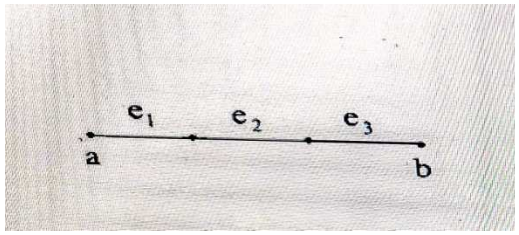

Fig.2

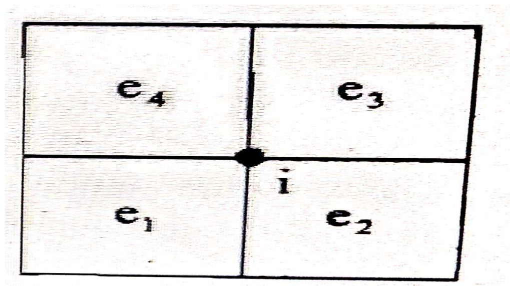

Fig.3

We exploitation the straight line elements in the one dimensional case, that is in solving ordinary differential equations (see in figure 1). In Fig. 1, the interval [a, b] is subdivided into three straight line elementse_1, e_2, e_3 . commonly, in two dimensions, we use the triangular or rectangular elements (see Figs.2, 3). In Fig.2, we have eight triangular elements numbered$e_1, e_2, e_3 \dots\dots\dots. e_8$. In Fig.3, we have four rectangular elements numbered e_1, e_2, e_3, e_4. The curved boundaries are handled in a natural manner.

In each of the finite element methods, within each finite element e, the solution is approximated by a function w that is continuous and defined in terms of the nodal values belonging to that element. At the boundaries of the elements, which are referred to as interfaces, it is crucial to ensure continuity and compatibility between neighboring elements. At interfaces between finite elements in the finite element method, a fundamental requirement is that the approximating function and its partial derivatives up to an order one less than the highest order derivative occurring in the partial differential equation or its variational form must be continuous. After ensuring continuity at the interfaces, the next step involves substituting the approximate solution w into the partial differential equation or its variational form.

Once substituted, the weighted residual method is applied. In the finite element method, an alternative approach involves substituting the solution w into the variational form of the partial differential equation. Solving this system provides the approximate solution of the partial differential equation at the nodal points within the domain R. For simpler networks or domains, it's worth noting that the difference equations derived by the finite difference and finite element methods can be identical. In this unit, as mentioned earlier, the focus is on solving Laplace and Poisson equations in two dimensions. For such problems, simple finite elements like triangular and rectangular elements can be used effectively. These elements simplify the discretization process and facilitate the assembly of the global system of equations, leading to efficient numerical solutions.

14.4 *TRIANGULAR ELEMENTS:-*

Absolutely, line segment elements are commonly used for solving ordinary differential equations, representing one-dimensional domains. When it comes to solving partial differential equations (PDEs) in two dimensions, triangular and rectangular elements are the fundamental choices. Assemblage of triangles can always represent a two dimensional domain of any shape. Normally, we use equilateral triangles

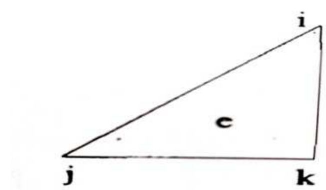

Fig.4

Suppose the triangular element of corners at $(x_i, y_i)(x_j, y_j)(x_k, y_k)$, taken in anticlockwise direction. This element is known as **three node triangle**. From the figure (4), we get

$$U^e(x, y) = a_1 + a_2 x + a_3 y \qquad \dots (1)$$

At the nodes, we have

$$U_i = a_1 + a_2 x_i + a_3 y_i$$
$$U_j = a_1 + a_2 x_j + a_3 y_j$$
$$U_k = a_1 + a_2 x_k + a_3 y_k$$

Now the solution of a_1, a_2 and a_3 , we obtain

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} U_i \\ U_j \\ U_k \end{bmatrix}$$

Using Cramer's rule, we obtain

$$a_1 = \frac{\Delta_1}{2\Delta}, a_2 = \frac{\Delta_2}{2\Delta}, a_3 = \frac{\Delta_3}{2\Delta}$$

Where Δ is the area of triangle.

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

$$= (x_i - x_k)(y_j - y_k) - (x_j - x_k)(y_i - y_k)$$

$$\Delta_1 = \begin{vmatrix} U_i & x_i & y_i \\ U_j & x_j & y_j \\ U_k & x_k & y_k \end{vmatrix}$$

$$= U_i(x_j y_k - x_k y_j) - U_j(x_i y_k - x_k y_i) + U_k(x_i y_j - x_j y_i)$$

$$\Delta_2 = \begin{vmatrix} 1 & U_i & y_i \\ 1 & U_j & y_j \\ 1 & U_k & y_k \end{vmatrix}$$

$$= U_i(y_j - y_k) - U_j(y_k - y_i) + U_k(y_i - y_j)$$

$$\Delta_3 = \begin{vmatrix} 1 & x_i & U_i \\ 1 & x_j & U_j \\ 1 & x_k & U_k \end{vmatrix}$$

$$= U_i(x_j - x_k) - U_j(x_k - x_i) + U_k(x_i - x_j)$$

Putting the values of a_1, a_2 and a_3 in equation (1), we obtain

$$U^e(x, y) = \frac{1}{2\Delta}(\Delta_1 + \Delta_2 x + \Delta_3 y)$$

$$= N_i^e(x, y)U_i + N_j^e(x, y)U_j + N_k^e(x, y)U_k \quad \dots (2)$$

where

$$N_i^e(x, y) = \frac{1}{2\Delta}[(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_i)y]$$

$$N_j^e(x, y) = \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_k - y_i)x + (x_i - x_k)y]$$

$$N_k^e(x, y) = \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_i - y_j)x + (x_j - x_i)y]$$

$$\left.\begin{aligned} N_i^e(x, y) &= \frac{1}{2\Delta}[(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_i)y] \\ N_j^e(x, y) &= \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_k - y_i)x + (x_i - x_k)y] \\ N_k^e(x, y) &= \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_i - y_j)x + (x_j - x_i)y] \end{aligned}\right\} \dots (3)$$

where N_i^e, N_j^e, N_k^e are known as **Shape Function** of the approximation.

Putting $x = x_i, y = y_i$ in equation (2), we obtain

$$U^e(x_i, y_i) = U_i$$

$$= N_i^e(x_i, y_i)U_i + N_j^e(x_i, y_i)U_j + N_k^e(x_i, y_i)U_k$$

Hence,

$$N_i^e(x_i, y_i) = 1$$
$$N_j^e(x_i, y_i) = 0$$
$$N_j^e(x_i, y_i) = 0$$

Similarly, putting $x = x_j, y = y_j$ and $x = x_k, y = y_k$ in equation (2), we get

$$N_i^e(x_i, y_i) = N_j^e(x_i, y_i) = N_k^e(x_i, y_i) = 1$$

and $\quad N_n^e(x_r, y_r) = 0$, for $n \neq r, r = i, j, k$

So we can write the equation (3), we get

$$U^e(x, y) = \left[N_i^e, N_j^e, N_k^e\right]\begin{bmatrix} U_i \\ U_j \\ U_k \end{bmatrix} = [N^e][U^e] \quad \dots (4)$$

where $[N^e] = \left[N_i^e, N_j^e, N_k^e\right]$ and $[U^e] = [U_i\ U_j\ U_k]^T$

If the domain contain is obtained by

$$U(x, y) = \sum_{e=1}^{k} U^e(x, y) = \sum_{e=1}^{k} [N^e][U^e]$$

So the elements that we are considering here are known as conforming elements.

14.5 *RECTANGULAR ELEMENTS:-*

Let the domain R divided into rectangular elements. Assume that the sides of the element are parallel to x and y axes respectively. Hence, we choose the piecewise polynomial in the form

$$U(x, y) = a_1 + a_2 x + a_3 y + a_4 \quad \dots (1)$$

Let i, j, k and n the rectangular element of corners P $(x_i, y_i), Q(x_j, y_j), R(x_k, y_k)$ and $S(x_k, y_k)$ taken in anticlockwise direction. This element is known as **three node triangle**.

From the figure (4), we obtain

$$U_i = a_1 + a_2 x_i + a_3 y_i + \ a_4 x_i y_i \quad \dots (2)$$
$$U_j = a_1 + a_2 x_j + a_3 y_j + a_4 x_j y_j \quad \dots (3)$$
$$U_k = a_1 + a_2 x_k + a_3 y_k + a_4 x_k y_k \quad \dots (4)$$
$$U_n = a_1 + a_2 x_i + a_3 y_n + a_4 x_k y_n \quad \dots (5)$$

Subtracting from (5) and (2), we have

$$(y_i - y_n) a_3 + x_i (y_i - y_n) a_4 = \ U_i - U_n$$
$$a_3 + a_4 x_i = \frac{U_i - U_n}{y_i - y_n} \quad \dots (6)$$

Subtracting from (4) and (3), we have

$$(y_i - y_n)a_3 + x_j(y_i - y_n)a_4 = U_j - U_k$$

$$a_3 + a_4x_j = \frac{U_j - U_k}{y_i - y_n} \qquad \dots (7)$$

From (6) and (7), we have

$$a_4(x_i - x_j) = \frac{U_i - U_n - U_j + U_k}{y_i - y_n}$$

$$a_4 = \frac{U_i - U_n - U_j + U_k}{(x_i - x_j)(y_i - y_n)}$$

Hence ,

$$a_3 = \frac{U_i - U_n}{(y_i - y_n)} - a_4x_i$$

From (3) and (2), we have

$$(x_i - x_j)a_2 + y_j(x_i - x_j)a_4 = U_i - U_j$$

$$a_3 + a_4x_j = \frac{U_j - U_k}{y_i - y_n} \qquad \dots (7)$$

Hence ,

$$a_3 = \frac{U_i - U_n}{(y_i - y_n)} - a_4y_i$$

Now from (2), we get

$$a_1 = U_i - a_2x_i - a_3y_i - a_4x_iy_i$$

Putting these above values in (1), then

$$U(x, y) = a_1 + a_2x + a_3y + a_4xy$$

$$= N_i^eU_i + N_j^eU_j + N_k^eU_k + N_n^eU_n$$

Where

$$N_i^e = \frac{(x - x_j)(y - y_n)}{(x_i - x_j)(y_i - y_n)}$$

$$N_j^e = \frac{(x - x_j)(y - y_n)}{(x_j - x_i)(y_i - y_n)}$$

$$N_k^e = \frac{(x - x_i)(y - y_i)}{(x_j - x_i)(y_n - y_i)}$$

$$N_n^e = \frac{(x - x_i)(y - y_i)}{(x_i - x_j)(y_n - y_i)}$$

Note: that the nodes are P(x_i, y_i), $Q(x_j, y_j)$, $R(x_k, y_k)$ and $S(x_k, y_k)$.

So

At the nodes P(x_i, y_i), we get

$N_i^e(x_i, y_i) = 1, N_j^e(x_i, y_i) = 0, \quad N_k^e(x_i, y_i) = 0 \quad N_n^e(x_i, y_i) = 0$

At the nodes Q(x_i, y_i), we have

$N_i^e(x_i, y_i) = 0, N_j^e(x_i, y_i) = 1, \quad N_k^e(x_i, y_i) = 0, \quad N_n^e(x_i, y_i) = 0$

At the nodes $R(x_i, y_i)$, we obtain
$N_i^e(x_i, y_i) = 0, N_j^e(x_i, y_i) = 0, \quad N_k^e(x_i, y_i) = 1, \quad N_n^e(x_i, y_i) = 0$
At the nodes $S(x_i, y_i)$, we obtain
$N_i^e(x_i, y_i) = 0, N_j^e(x_i, y_i) = 0, \quad N_k^e(x_i, y_i) = 0, \quad N_n^e(x_i, y_i) = 1$
Therefore, the shape functions have the value 1 at the node where it is explained and have the value 0 at all the other nodes.
Hence, if the sides of the rectangle are not parallel to the axes, then shape functions satisfy the condition

$$N_r^e(x_s, y_s) = 1 \quad or \quad r = s$$
$$N_r^e(x_s, y_s) = 0 \quad or \quad r \neq s$$

Let R be a two-dimensional domain. Suppose a typical node , known as an apex node. The nodes marked in Fig.2 and 3 are apex nodes. In Fig.2, the apex i is common to six triangular elements. In Fig.3, the apex i is common to four rectangular elements. The piecewise approximating function $U(x, y)$ over the whole domain R can be given below

$$U(x, y) = \sum_{i=1}^{n} N_i^e(x, y)\, U_i$$

where M is the number of nodes contained in R , N_i are the interpolating functions and U_i are the values at the nodes.
The focus shifts to applying finite element methods for solving Laplace and Poisson equations within a two-dimensional domain. Finite element methods offer flexibility in approaching these equations, either by extremizing the variational form of the partial differential equation or by directly employing a weighted residual method. Galerkin's method, a popular approach, falls under the weighted residual approach category. The subsequent section delves into Galerkin's method, culminating in the derivation of the finite element Galerkin's method. While the discussion primarily revolves around two-dimensional boundary value problems, it's emphasized that the method holds potential for generalization to any dimension. This groundwork sets the stage for a deeper exploration of finite element techniques in solving differential equations within various spatial domains.

14.6 *GALERKIN METHOD:-*

Let the boundary problem is is given as

$$L(u) = f(x, y); \;\; x, y \in R \qquad \ldots (1)$$

$$B_i[u] = f(x,y); \;\; x,y \in R \qquad \dots (2)$$

where L is the second order linear differential operator, r is the boundary of R and B_i are the boundary conditions. We acquire the solution of the partial differential equation in the form is given as

$$U(x,y) = w(x,y,a_1,a_2 \dots \dots a_n) = \phi_0(x,y) + \sum_{i=1}^{n} a_i\phi_i(x,y)$$

which depends on the parameters $a_1, a_2 \dots \dots a_n$ am and $\phi_i(x,y)$ are basis functions.

So $\phi_i(x,y)$ satisfies the inhomogeneous boundary conditions and $\phi_i(x,y)$, $i = 1,2 \dots \dots$ satisfy the homogeneous boundary condition given below

$$B_i = g_i, B_i[\phi_j] = 0, j = 12, \dots, n$$

Thr error is

$$E(x,y,a) = L[w(x,y,a)] - f(x,y)$$

where $a = [a_1, a_2 \dots . a_n]^T$. In the Galerkin method, the error is minimized by making it orthogonal to the chosen set of functions over the entire domain $\phi_i(x,y), i = 1,2 \dots \dots, n$. In other words,

$$\iint_R E(x,y,a)\, \phi_j(x,y)\, dxdy = 0, j = 1,2, \dots, n$$

This obtain $n \times n$ system of equations for the solution of $a_1, a_2 \dots . a_n$.
Let's apply the method to solve the boundary value problem defined by Eqns. (16) and (17). Assuming k as the number of nodes in an element, we express the approximation within an element e as (remainder of the equation).

$$U(x,y) = \sum_{i=1}^{k} N_i U_i = N^e U^e$$

where $N^e = [N_1, N_2, \dots . N_k]$ and $U^e = [U_1, U_2, \dots . U_k]^T$.The residual is $E = L(U) - f$.
Here, we choose the weight function as N_i. Therefore

$$\iint_R [L(U) - f] N_i dxdy = 0, i = 1,2,3 \dots, k \;\; \dots (3)$$

Since

$$\iint_{e_j} [L(U) - f]N_i dxdy = 0, i = 1,2,3 \ldots, k, j = 1,2,3 \ldots, n$$

where N_i represents the shape functions within the element and U_i denotes the values of the variable at the nodes within the element.

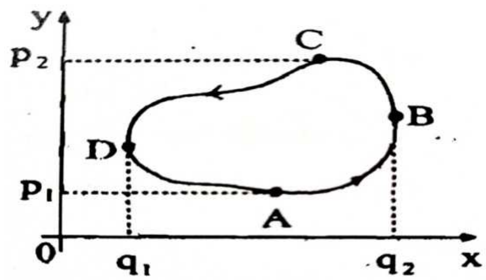

Fig.5

Let's apply this method to solve the Dirichlet boundary value problem for the Poisson equation, which is described by

$$LU + G(x, y) = U_{xx} + U_{yy} + G(x, y) = 0 \text{ in } R \quad \ldots (4)$$

$$U(x, y) = 0 \text{ on } \Gamma$$

Let R contain n elements, each element with k nodes. We express the approximation across the entire domain as

$$U(x, y) = \sum_{i=1}^{n} N_i (x, y)U_i \qquad \ldots (5)$$

Putting $U(x, y)$ in the Eqn.(4) and using Eqn.(3), we get

$$\sum_{i=1}^{n}\left[\iint_R \{(N_i)_{xx} + (N_i)_{yy}\}N_j \, dxdy\right]U_i + \iint_R G(x, y)N_j \, dxdy = 0, j = 1,2,3 \ldots, n \qquad \ldots (6)$$

Let R be the domain as obtained in Fig.5.

The curve CDA: $x = h_1(y)$.
The curve ABC:$x = h_2(y)$.
Therefore, R is assumed as

$$\boldsymbol{R}: \quad q_1 \le y \le q_2, g_1(x) \le y \le g_2(x) \qquad \ldots (7)$$

we can also state that R is enclosed by the curves DAB and BCD.
The curve DAB : $y = g_1(y)$..
Equation of curve BCD : $y = g_2(y)$.
Therefore, R is also described as

$$\boldsymbol{R}: \quad q_1 \le x \le q_2, g_1(x) \le x \le g_2(x) \qquad \dots (8)$$

by integrating the first term of Eqn.(6) and utilizing the definition of R as provided in Eqn.(8), we obtain (refer to Fig. 5).

$$\int_R\int (N_i)_{xx}\, N_j\, dxdy = \int_{y=p_1(x)}^{p_2}\int_{x=h_1(y)}^{h_2(y)} (N_i)_{xx}\ N_j dxdy$$

$$= \int_{p_1}^{p_2}\int_{x=h_1(y)}^{h_2(y)} \left[(N_i)_x\, N_j\right]_{x=h_1(y)}^{h_2(y)} dy - \int_R\int (N_i)_x\left(N_j\right)_x dxdy$$

$$= \oint_\Gamma (N_i)_x\ N_j d\Gamma - \int_R\int (N_i)_{xx}\, N_j\, dxdy \qquad \dots (9)$$

The first integral represents integration along the boundary r (a line integral). Likewise, the definition of R from Eqn.(8), the second term of Eqn.(6) yields:

$$\int_R\int (N_i)_{xx}\, N_j\, dxdy = \oint_\Gamma (N_i)_y\ N_j d\Gamma - \int_R\int (N_i)_y\left(N_j\right)_y dxdy \quad \dots (10)$$

The contributions of the first term on the right-hand side of Eqns (9) and (10) are zero for all elements inside. Only elements with a portion of the boundary as sides have contributions from the natural boundary conditions. When no natural boundary conditions are specified, these contributions are considered zero. For the boundary value problems under study, we set these terms to zero. Thus, from Eqns. (6), (9), and (10), we derive:

$$\sum_{i=1}^{n}\left[\int_R\int \left\{(N_i)_x\left(N_j\right)_x + (N_i)_y\left(N_j\right)_y\right\} dxdy\right] U_i = \int_R\int G N_j\, dxdy = 0, j = 1,2,3\dots,n \qquad \dots (11)$$

For the Laplace equation $\nabla^2 U = 0$, $G(x,y)$ =0 on the right-hand side of Eqn. (11). The integrals are computed for each element, and then the element equations are combined. The equation at each node i $(or\ apex\ i)$ is constructed from the contributions of all elements sharing it. The coefficient matrix of the resulting system forms a band matrix.

EXAMPLE: Find the solution of the boundary value problem

$$\nabla^2 U = x^2 + y^2, 0 \le x \le 1, 0 \le y \le 1$$
$$U = \frac{1}{12}\left(x^4 + y^4\right) \text{ on the boundary}$$

using the Galerkin method with (i) triangular elements, (iii) rectangular elements and one internal node $(h = 1/2)$.

SOLUTION:

i. **Triangular Elements:** The mesh is obtained in Fig 6. At the six triangular elements e_1, e_2, e_3, e_4 and e_5. So the boundary values are

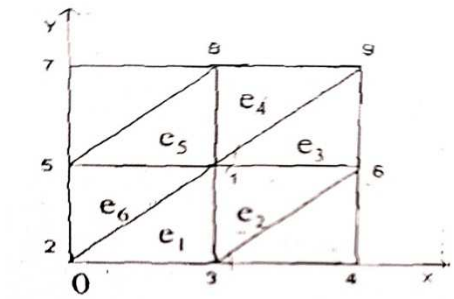

Fig.7 Triangular elements

$$U_2 = U(0,0), U_3 = U\left(\frac{1}{2}, 0\right) = \frac{1}{192} = U_5$$
$$U_2 = U\left(1, \frac{1}{2}\right) = \frac{17}{192} = U_8, U_9 = U(1,1) = \frac{1}{6}$$

Comparing these equations with (4), we obtain
$$G(x, y) = -(x^2 + y^2)$$

Now from (11), we get
$$\sum_{i=1}^{n}\left[\int\int_R \left\{(N_n)_x (N_i)_x + (N_n)_y \left(N_j\right)_y\right\} dxdy\right] U_n + \int\int_R (x^2 + y^2)\, dxdy$$
$$= 0, j = 1,2,3 \dots, n \qquad \dots (1)$$

Let's determine the contribution of each element. The apex node is consistently denoted by i, while the remaining nodes are labeled j and k in an anti-clockwise direction. The shape functions N_i, N_j and N_k are written as

$$N_i = \frac{1}{2\Delta}\left[\left(x_j y_k - x_k y_j\right) + \left(y_j - y_k\right)x + (x_k - x_i)y\right]$$

$$N_j = \frac{1}{2\Delta}\left[(x_k y_i - x_i y_k) + (y_k - y_i)x + (x_i - x_k)y\right]$$

$$N_k = \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_i - y_j)x + (x_j - x_i)y]$$

Where Δ= area of triangle

Now

Element e_1: $i = \left(\frac{1}{2}, \frac{1}{2}\right), j = (0,0), k = \left(\frac{1}{2}, 0\right), \Delta = \frac{1}{8}$

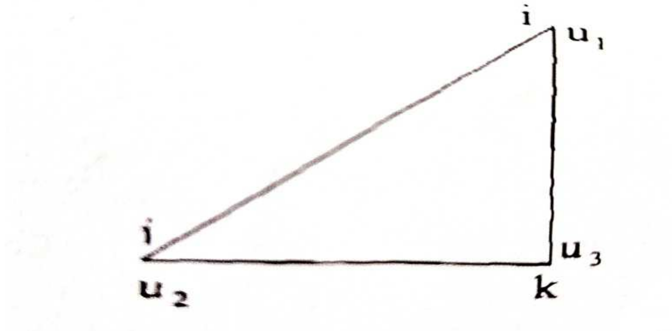

Fig. 8

$$N_i = 4\left[\frac{1}{2}, y\right] = 2y, \qquad N_j = 4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right) x\right] = 1 - 2x$$

$$N_k = 4\left[\frac{1}{2}x - \frac{1}{2} y\right] = 2(x - y)$$

Now we obtain

$$(N_i)_x = 0, (N_i)_y = 2, (N_j)_x = -2, (N_j)_y = 0$$
$$(N_k)_x = 2, (N_k)_y = -2$$

So putting these values in (1), we given

$$\left[\iint_{e_1} 4dxdy\right]U_1 + \left[\iint_{e_1} (0)dxdy\right]U_2 + \left[\iint_{e_1} -4dxdy\right]U_3$$
$$+\left[\iint_{e_1} (x^2 + y^2)(2y)dxdy\right]$$

Now

$$\iint_{e_1} f(x,y)dxdy = \int_{x=0}^{1,2}\left[\int_{y=0}^{x} f(x,y)\,dy\right]dx$$

Now the integrating it, we have

$$4\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)U_3 + 2\left(\frac{3}{4}\right)\left(\frac{1}{160}\right) = \frac{1}{2}U_1 - \frac{1}{2}\left(\frac{1}{192}\right) + \frac{3}{320}$$
$$= \frac{1}{2}U_1 + \frac{13}{1920}$$

Element e_2:

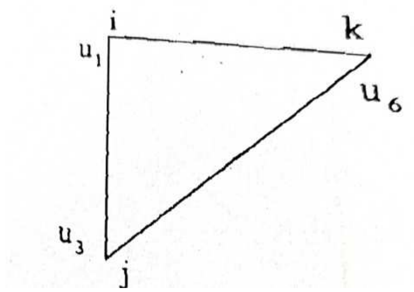

Fig.9

$i = \left(\frac{1}{2},\frac{1}{2}\right), j = \left(\frac{1}{2},0\right), k = \left(1,\ \frac{1}{2}\right), \Delta = \frac{1}{8}$ (From fig.9)

$$N_i = 4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)x + \left(1 - \frac{1}{2}\right)y\right] = 1 - 2x + 2y$$

$$N_j = 4\left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x + \left(\frac{1}{2} - 1\right)y\right] = 1 - 2y$$

$$N_k = 4\left[-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)x\right] = -1 + 2x$$

We obtain

$$(N_i)_x = -2, (N_i)_y = 2, \left(N_j\right)_x = 0, \left(N_j\right)_y = -2$$

$$(N_k)_x = 2, (N_k)_y = 0$$

Putting these values in equation (1), we get

$$\left[\iint_{e_1} \{(-2)(2) + (2)(2)\}dxdy\right]U_1 + \left[\iint_{e_1} (-2)(2)dxdy\right]U_3$$
$$+ \left[\iint_{e_2} (2)(-2)dxdy\right]U_6$$
$$+ \left[\iint_{e_2} (x^2 + y^2)(1 - 2x + 2y)dxdy\right]$$

Now

$$\int_{e_2}\int f(x,y)dxdy = \int_{x=1,2}^{1}\left[\int_{y=x-1/2}^{1/2} f(x,y)\,dy\right]dx$$

Now the integrating it, we have

$$8\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)\left(\frac{1}{192}\right) + 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right) + \frac{11}{480} = U_1 - \frac{23}{960}$$

Element e_3:

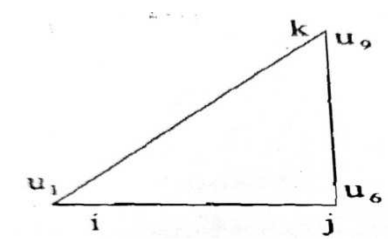

Fig.10

$i = \left(1, \frac{1}{2}\right), k = (1,1), \Delta = \frac{1}{8}$ (From fig.10)

$$N_i = 4\left[1 - \left(\frac{1}{2}\right)x + \left(\frac{1}{2} - 1\right)x\right] = 2(1-x)$$

$$N_j = 4\left[\left(1 - \frac{1}{2}\right)x + \left(1 - \frac{1}{2}\right)y\right] = 2(x-y)$$

$$N_k = 4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \frac{1}{2} + \left(1 - \frac{1}{2}\right)y\right] = -1 + 2y$$

We obtain

$$(N_i)_x = -2, (N_i)_y = 0, \left(N_j\right)_x = 2, \left(N_j\right)_y = -2$$
$$(N_k)_x = 0, (N_k)_y = 2$$

So putting these values in (1), we given

$$\left[\int_{e_3}\int 4dxdy\right]U_1 + \left[\int_{e_3}\int -4dxdy\right]U_6 + (0)U_9 + \left[\int_{e_3}\int (x^2+y^2)(1-x)dxdy\right]$$

Now, $\int_{e_3}\int f(x,y)dxdy = \int_{x=1,2}^{1}\left[\int_{y=1,2}^{x} f(x,y)\,dy\right]dx$

Now the integrating it, we have

$$4\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)U_6 + \left(\frac{39}{960}\right) = \frac{1}{2}U_1 - \frac{1}{2}\left(\frac{17}{192}\right) + \frac{39}{960} = \frac{1}{2}U_1 - \frac{7}{1920}$$

Element e_4:

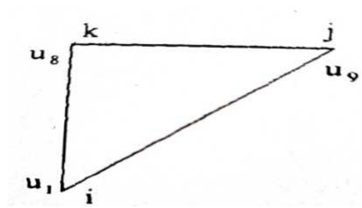

Fig.11

$i = \left(\frac{1}{2},\frac{1}{2}\right), k = (1,1),\ k = \left(\frac{1}{2},1\right), \Delta = \frac{1}{8}$ (From fig.11)

$$N_i = 2(1-y)$$

$$N_j = -1 + 2x$$

$$N_k = 2(-x+y)$$

We obtain

$$(N_i)_x = 0, (N_i)_y = -2, \left(N_j\right)_x = 2, \left(N_j\right)_y = 0$$

$$(N_k)_x = -2, (N_k)_y = 2$$

So putting these values in (1), we have

$$\left[\iint_{e_4} 4dxdy\right]U_1 + \left[\iint_{e_3} -4dxdy\right]U_8(0)U_9 + \left[\iint_{e_3} (x^2+y^2)(1-y)dxdy\right]$$

Now

$$\iint_{e_4} f(x,y)dxdy = \int_{x=1/2}^{1}\left[\int_{y=x}^{1} f(x,y)\,dy\right]dx$$

Now the integrating it, we have

$$4\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right) + \left(\frac{39}{960}\right) = \frac{1}{2}U_1 - \frac{7}{1920}$$

Element e_5:

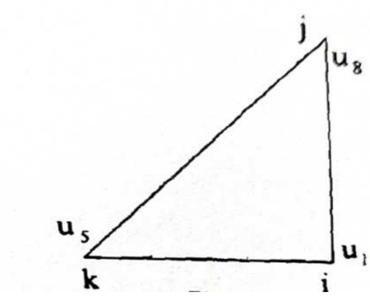

Fig.12

$i = \left(\frac{1}{2}, \frac{1}{2}\right), i = \left(\frac{1}{2}, 1\right), k = \left(0, \ \frac{1}{2}\right), \Delta = \frac{1}{8}$ (From fig.12)

We obtain

$$(N_i)_x = 2, (N_i)_y = -2, \left(N_j\right)_x = 0, \left(N_j\right)_y = 2$$

$$(N_k)_x = -2, (N_k)_y = 0$$

So putting these values in (1), we given

$$\left[\iint_{e_5} 8dxdy\right]U_1 + \left[\iint_{e_5} -4dxdy\right]U_8 + \left[\iint_{e_5} -4dxdy\right]U_5 + \left[\iint_{e_5} (x^2+y^2)(1-2x+2y)dxdy\right]$$

Now

$$\iint_{e_5} f(x,y)dxdy = \int_{x=0}^{1/2}\left[\int_{y=1/2}^{(1+2x)/2} f(x,y)\,dy\right]dx$$

The integrating it, we have

$$8\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right) + 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right) + \left(\frac{11}{480}\right) = U_1 - \frac{23}{960}$$

Element e_6:

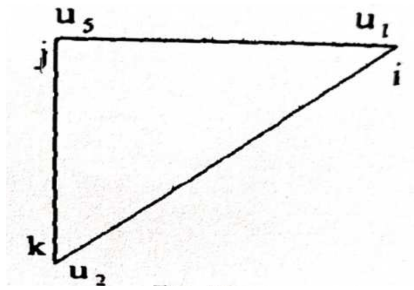

Fig.13

$$i = \left(\frac{1}{2}, \frac{1}{2}\right), j = \left(0, \ \frac{1}{2}\right), k = (0,0), \Delta = \frac{1}{8} \text{(From fig. 13)}$$

$$N_i = 2x$$

$$N_j = -2x + 2y$$

$$N_k = 1 - 2y$$

We obtain

$$(N_i)_x = 2, (N_i)_y = 0, \left(N_j\right)_x = -2, \left(N_j\right)_y = 2$$

$$(N_k)_x = 0, (N_k)_y = -2$$

Putting these values in equation (1), we have

$$\left[\iint_{e_6} 4dxdy\right] U_1 + \left[\iint_{e_6} -4dxdy\right] U_{5+}(0)U_2 \left[\iint_{e_6} (x^2 + y^2)2xdxdy\right]$$

Now

$$\iint_{e_6} f(x,y)dxdy = \int_{x=0}^{1/2} \left[\int_{y=x}^{1/2} f(x,y)\, dy\right] dx$$

Now the integrating it, we have

$$4\left(\frac{1}{8}\right)U_1 - 4\left(\frac{1}{8}\right)\left(\frac{1}{192}\right) + \left(\frac{9}{960}\right) = \frac{1}{2}U_1 + \frac{13}{1920}$$

Adding all the contributions yields the node 1's difference equation.

$$\left(\frac{1}{2}+1+\frac{1}{2}+1+\frac{1}{2}\right)U_1 + \left(\frac{13}{1920}-\frac{23}{960}-\frac{7}{1920}-\frac{7}{1920}-\frac{23}{960}+\frac{13}{1920}\right)=0$$

$$4U_1-\frac{1}{24}=0$$

Hence , $U_1=\frac{1}{96}$

The function $u(x,y)=\frac{(x^4+y^4)}{12}$ satisfies both the differential equation and the boundary condition for the given problem, making it the exact solution. Interestingly, this exact solution coincides with the finite element solution.

ii. **Rectangular elements:**

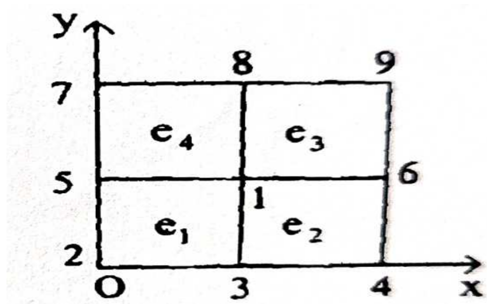

Fig.14

In Fig.14, the mesh shows four rectangular elements contributing at 1. Boundary values remain consistent, with an additional value $U_4=U(1,0)=\frac{1}{12}=U_7$. The apex node is labeled as i, while the remaining nodes j,k and m are numbered counterclockwise. Shape functions The shape functions N_i,N_j,N_k and N_n are written as(from equation 14)

$$N_i=\frac{(x-x_j)(y-y_n)}{(x_i-x_j)(y_i-y_n)}$$

$$N_j=\frac{(x-x_j)(y-y_n)}{(x_j-x_i)(y_i-y_n)}$$

$$N_k=\frac{(x-x_i)(y-y_i)}{(x_j-x_i)(y_n-y_i)}$$

$$N_n=\frac{(x-x_i)(y-y_i)}{(x_i-x_j)(y_n-y_i)}$$

Element e_1: $i = \left(\frac{1}{2}, \frac{1}{2}\right), i = \left(0, \ \frac{1}{2}\right), k = (0, 0), \ n = \left(\frac{1}{2}, 0\right) \ \Delta = \frac{1}{8}$ (From fig.15)

We obtain

$$N_i = \frac{(x-0)(y-0)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = 4xy$$

$$N_j = \frac{\left(x-\frac{1}{2}\right)(y-0)}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)} = -2(2x-1)y$$

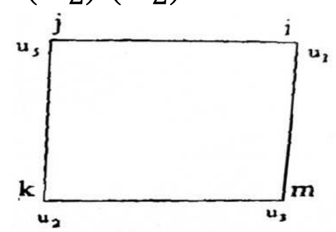

Fig.15

$$N_k = \frac{\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)} = (2x-1)(2y-1)$$

$$N_n = \frac{(x-0)\left(y-\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)} -= -2(2y-1)x$$

Substituting these values in equation (1), we have

$$\left[\iint_{e_1} 16(x^2+y^2)dxdy\right]U_1$$
$$+\left[\iint_{e_1} \{(-4y)(4y) - 2(2x-1)4x\}dxdy\right]U_5$$
$$+\left[\iint_{e_1} \{2(2y-1)4y + 2(2x-1)2x\}dxdy\right]U_2$$
$$+\left[\iint_{e_1} \{2(2y-1)4y - 4x4y\}dxdy\right]U_3$$
$$+\iint_{e_1} (x^2+y^2)(4xy)dxdy$$

Now

$$\int\int_{e_1} f(x,y)dxdy = \int_{x=0}^{1/2}\left[\int_{y=0}^{1/2} f(x,y)\,dydx\right]$$

The integrating it, we have

$$16\left(\frac{1}{24}\right)U_1 - \left(\frac{1}{6}\right)\left(\frac{1}{192}\right) - 0 - \left(\frac{1}{6}\right)\left(\frac{1}{192}\right) + \left(\frac{1}{64}\right) = \frac{2}{3}U_1 + \frac{1}{72}$$

Element e_2:

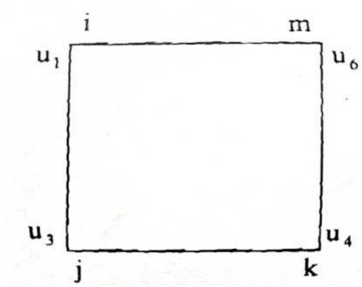

Fig.16

$i = \left(\frac{1}{2},\frac{1}{2}\right), i = \left(\frac{1}{2},0\right), k = (1,0),\ n = \left(1,\frac{1}{2}\right)\ \Delta = \frac{1}{8}$ (From fig.16)

$$N_i = 4(x-1)y,\ \ N_j = 2(x-1)(2y-1)$$
$$N_k = -(2x-1)(2y-1),\ \ N_n = 2(2x-1)y$$

From (1), we get

$$\left[\int\int_{e_2}\{(-4y)^2 + 16(x-1)^2\}\,dxdy\right]U_1 + \left[\int\int_{e_2}\{-2(2y-1)(4y) + 4(x-1)(-4)(x - 1)\}dxdy\right]U_3 + \left[\int\int_{e_2}\{-2(2y-1)(-4y) - 2(2x-1)(-4)(x - 1)\}dxdy\right]U_4 + \int\int_{e_2}(x^2+y^2)(-4)(x-1)ydxdy$$

$$\int\int_{e_2} f(x,y)dxdy = \int_{x=1/2}^{1}\left[\left(\int_{y=0}^{1/2} f(x,y)\,dy\right)dx\right]$$

The integrating it, we have

$$\left(\frac{2}{3}\right)U_1 - \left(\frac{1}{6}\right)\left(\frac{1}{192}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{12}\right) - \frac{1}{6}\left(\frac{17}{192}\right) + \frac{7}{192} = \frac{2}{3}U_1 - \frac{1}{144}$$

Element e_3:

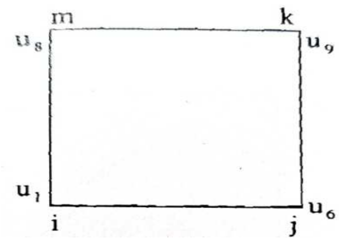

Fig.17

$i = \left(\frac{1}{2},\frac{1}{2}\right), i = \left(1,\frac{1}{2}\right), k = (1,1),\; n = \left(\frac{1}{2},1\right)$ (From fig.17)

$$N_i = 4(x-1)(y-1),\;\; N_j = 2(x-1)(y-1)$$

$$N_k = -(2x-1)(2y-1),\;\; N_n = -2(x-1)(2y-1)$$

Equation (1), we get

$$\left[\int\int_{e_3}\{16(y-1)^2 + 16(x-1)^2\}\,dxdy\right]U_1$$

$$+\left[\int\int_{e_3}\{-1616(y-1)^2 - 2(2x-1)(x-1)(4)(x - 1)\}dxdy\right]U_6$$

$$+\left[\int\int_{e_3}\{2(2y-1)(4)(y-1) + 2(2x-1)(4)(x - 1)\}dxdy\right]U_9$$

$$+\left[\int\int_{e_3}\{-2(2y-1)(4)(y-1) - 4(2x-1)(4)(x - 1)\}dxdy\right]U_8 + \int\int_{e_3}(x^2+y^2)(4)(x-1)(y-1)dxdy$$

The integrating it, we have

$$\left(\frac{2}{3}\right)U_1 - \left(\frac{1}{6}\right)\left(\frac{17}{192}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) - \frac{1}{6}\left(\frac{17}{192}\right) + \frac{11}{192} = \frac{2}{3}U_1 - \frac{1}{36}$$

Elemente_4:

$i = \left(\frac{1}{2}, \frac{1}{2}\right), i = \left(\frac{1}{2}, 1\right), k = (0, 1),\ n = \left(0, \frac{1}{2}\right)$ (From fig.18

$$N_i = -4(y-1)(y-1),\ \ N_j = (2y-1)$$

$$N_k = -(2x-1)(2y-1),\ \ N_n = -2(2x-1)(y-1)$$

Equation (1), we have

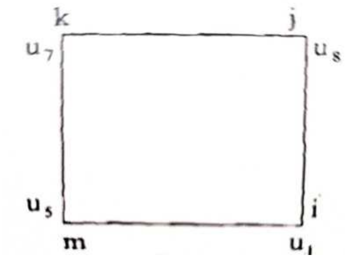

Fig.18

$$\left[\iint_{e_4} \{16(y-1)^2 + 16(x)^2\}\, dxdy\right] U_1$$

$$+ \left[\iint_{e_4} \{2(2y-1)(-4)(y-1) + 2(2x-1)(4x)(-4x)\} dxdy\right] U_8$$

$$+ \left[\iint_{e_4} \{-2(2y-1)(-4)(y-1) - 2(2x-1)(-4x)\} dxdy\right] U_7$$

$$+ \left[\iint_{e_4} \{-4(y-1)(-4)(y-1) - 2(2x-1)(-4x)\} dxdy\right] U_5$$

$$+ \iint_{e_4} (x^2 + y^2)(-4)(x)(y-1) dxdy$$

The integrating it, we have

$$\left(\frac{2}{3}\right)U_1 - \left(\frac{1}{6}\right)\left(\frac{17}{192}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{12}\right) - \frac{1}{6}\left(\frac{1}{192}\right) + \frac{7}{192} = \frac{2}{3}U_1 - \frac{1}{144}$$

Adding all contributions, we get

$$\left(\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)U_1 + \frac{1}{72} - \frac{1}{144} - \frac{1}{36} - \frac{1}{144} = 0$$

$$\frac{8}{3}U_1 - \frac{1}{36} = 0$$

We have $U_1 = \frac{1}{96}$ is required solution.

In Example 1, coincidentally, the finite element solution matches the exact solution. However, in practice, achieving accurate solutions often requires a large number of elements, whether triangular or rectangular. This increases computational complexity. Typically, we start with a certain number of elements, M, then gradually increase it, monitoring convergence of solution values at nodes to determine when to stop computations. Finite element methods can be applied to extremize the variational form of partial differential equations. Here, we present the variational form of Laplace's and Poisson's equations.

14.7 *VARIATIONAL FORMS:-*

Let us suppose a functional in one independent variable x in the form is given by

$$V(u) = \int_{x_1}^{x_2} F(x, u, u_x, u_{xx})\, dx \qquad \dots(1)$$

The objective is to find $u(x)$, referred to as an extremal, to maximize the functional in Eqn.(1). According to the theory of variations, it can be demonstrated that the $u(x)$ which maximizes (1) also satisfies the Euler-Lagrange equation.

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u_{xx}}\right) = 0 \qquad \dots(2)$$

Hence, solving the partial differential Eqn.(2) maximizes the functional in Eqn.(1), and the function maximizing (1) is the solution of Eqn.(2).

Geometric analysis helps determine if it minimizes Eqn.(1) or not. Now, let's examine a functional with two independent variables, x and y:

$$V(u) = \iint_R F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dA \quad \dots (3)$$

The corresponding Euler-Lagrange equation can be obtained as given below

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial u_{xy}}\right) = 0 \quad \dots (4)$$

The function being varied must have continuous first-order derivatives and satisfy certain boundary conditions.

$$\left[\frac{\partial F}{\partial u_x} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_{xx}}\right)\right]_{x_1}^{x_2} = 0 \quad \text{and} \quad \left[\left(\frac{\partial F}{\partial u_{xx}}\right)\right]_{x_1}^{x_2} = 0$$

These conditions referred to as the natural boundary conditions of the problem are constraints that arise naturally from the physical or mathematical context of the problem.

EXAMPLE: Consider a functional as

$$V(u) = \iint_R \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2uf(x, y)\right] dA \quad \dots (5)$$

Using Equation (4), the **Euler-Lagrange equation** can be expressed as:

$$-2f - \frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\right] - \frac{\partial}{\partial y}\left[2\frac{\partial u}{\partial y}\right] = 0$$

$$u_{xx} + u_{yy} + f(x, y) = 0 \quad \dots (6)$$

We obtain a special case of the **Poisson equation known as Laplace's equation**. If the source term f is set to zero, then this equation can be written as

$$V(u) = \iint_R \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dA \quad \dots (7)$$

and it is known as Euler-Lagrange equation is the Laplace equation $u_{xx} + u_{yy} = 0$ .Eqns.() and (38) are respectively called the variation formulations of the Laplace equation $u_{xx} + u_{yy} = 0$ and Poisson's Equation $u_{xx} + u_{yy} + f(x, y) = 0$.

Note that the variational formulation of a partial differential equation typically involves lower-order partial derivatives, which can simplify solving the problem. However, it's often challenging to derive the variational form directly from the equation. For boundary value problems, if the differential operator is self-adjoint, both the classical variational principle and Galerkin's method yield the same matrix system. This equivalence holds for the Laplace and Poisson equations. However, if the operator isn't self-adjoint, the difference equations from the two methods differ. Since this unit focuses on solving the Laplace and Poisson equations, Galerkin's method suffices for our purposes.

SELF CHECK QUESTIONS

1. What is the Finite Element Method (FEM)?
2. What are the key steps involved in applying the Finite Element Method?
3. What is the purpose of meshing in the Finite Element Method?
4. What is the role of shape functions in the Finite Element Method?
5. How are boundary conditions applied in the Finite Element Method?
6. What are the advantages of the Finite Element Method?
7. What are some limitations of the Finite Element Method?

14.8 *SUMMARY:-*

In this unit, we've covered the following key points:

- Finite Element Methods (FEM) solves partial differential equations by dividing the domain into finite elements.
- Different types of elements (line, triangular, rectangular) are used based on the dimensionality of the problem.
- Solutions are approximated using piecewise continuous polynomials defined by nodal values.
- Difference equations can be derived using variational principles or weighted residual methods.

- The Finite Element Galerkin method, a weighted residual method, doesn't require the variational form.
- For self-adjoint partial differential equations, both variational principles and Galerkin's method yield the same matrix system.

Overall the Finite Element Method (FEM) is a numerical technique used to solve partial differential equations by dividing the problem domain into smaller elements. It approximates the solution within each element using piecewise functions called shape functions, allowing for accurate representation of complex geometries and boundary conditions. The method involves discretizing the domain, formulating element equations, assembling a global system of equations, applying boundary conditions, solving the system, and post-processing to obtain desired results. FEM is versatile; handling various types of problems, but requires careful meshing and can be computationally intensive for large systems.

14.9 *GLOSSARY:-*

- **Finite Element Method (FEM):** A numerical technique for solving partial differential equations by dividing the domain into smaller, simpler elements.
- **Finite Element:** A small sub-domain within the problem domain, often represented by simple geometrical shapes like triangles or rectangles.
- **Shape Function:** Functions used to approximate the behavior of the unknown field within each finite element.
- **Mesh:** The discretization of the problem domain into finite elements.
- **Nodal Values:** Values of the unknown field at the vertices of the finite elements.
- **Variational Principle:** A mathematical principle used to derive the weak form of partial differential equations, which is often utilized in FEM.
- **Weighted Residual Method:** A technique for deriving difference equations in FEM by minimizing the residual error in a weighted sense.
- **Galerkin Method:** A specific weighted residual method where the trial functions are chosen from the same function space as the residual.

- **Boundary Conditions:** Constraints imposed on the solution at the boundaries of the problem domain.
- **Assembly:** The process of combining local element equations into a global system of equations.
- **Post-processing:** The analysis of numerical results obtained from solving the system of equations.
- **Self-adjoint:** A property of a differential operator where the adjoint operator is the same as the original operator.

These terms provide a basic understanding of the concepts and terminology used in Finite Element Method.

14.10 *REFERENCES:-*

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- David V. Hutton (2004), Fundamentals of Finite Element Analysis.

14.11 *SUGGESTED READING:-*

- https://egyankosh.ac.in/bitstream/123456789/12562/1/Unit-12.pdf
- S.S.Sastry (Fifth edition 2012) Introductory Methods of Numerical Analysis.
- https://www.lkouniv.ac.in/site/writereaddata/siteContent/202004032250572068siddharth_bhatt_engg_Numerical_Solution_of_Partial_Differential_Equations.pdf

14.12 *TERMINAL QUESTIONS:-*

(TQ-1): By Galerkin's method to solve the boundary value problems.

i. $\frac{d^2y}{dx^2} + y = x^2,$ $\qquad y(0) = y(1) = 0$

ii. $\frac{d^2y}{dx^2} - 64y + 10 = x^2, \qquad y(0) = y(1) = 0$

(TQ-2): Find the boundary value problem

$$\nabla^2 u = 4, 0 \le x \le 1, 0 \le y \le 1$$

$$u = x^2 + y^2, \quad \text{on the boundary}$$

Using the Galerkin method with

(i) Rectangular elements,
(ii) Triangular elements and one internal node $(h = 1/2)$

(TQ-3): Find the boundary value problem

$$\nabla^2 u = x^2 + 2y^2, 0 \le x \le 1, 0 \le y \le 1$$

$$u = \frac{1}{12}(x^4 + 2y^4) \text{ on the boundary.}$$

Using the Galerkin method with

(i) Rectangular elements,
(ii) Triangular elements and one internal node $(h = 1/2)$

14.13 *ANSWERS:-*

SELF CHECK ANSWERS

1. The Finite Element Method (FEM) is a numerical technique used for solving partial differential equations by dividing the problem domain into smaller, simpler elements, where the solution is approximated by piecewise functions over each element.
2. The key steps in applying the Finite Element Method include:

- Discretization of the domain into elements.
- Formulation of element equations based on the governing differential equations.
- Assembly of the global system of equations from the element equations.
- Application of boundary conditions.
- Solution of the resulting system of equations.
- Post-processing to obtain desired quantities of interest.

3. Meshing is the process of dividing the problem domain into smaller elements. It helps in simplifying the problem, as each

element can be analyzed individually, and also allows for accurate representation of complex geometries.

4. Shape functions are used to approximate the behavior of the unknown field within each finite element. They define the variation of the field within an element in terms of nodal values, facilitating the interpolation of values at any point within the element.
5. Boundary conditions are typically applied by modifying the global system of equations to incorporate the known values or constraints at the boundaries of the domain. This is done during the assembly of the system of equations, ensuring that the solution satisfies the prescribed boundary conditions.
6. Some advantages of the Finite Element Method include its ability to handle complex geometries, its versatility in dealing with various types of boundary conditions and material properties, and its capability to provide accurate solutions for a wide range of engineering problems.
7. Limitations of the Finite Element Method include the need for careful meshing, particularly in regions of high gradients or singularities, the computational cost associated with solving large systems of equations, and the potential for numerical instabilities if not applied correctly.

TERMINAL ANSWERS

(TQ-1): i. $v(x) = -\frac{10}{123}x(1-x) - \frac{7}{41}x^2(1-x)$, ii. $v(x) = \frac{25}{37}x(1-x)$

(TQ-2): $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

(TQ-3): $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{64}$

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University, Haldwani, Nainital-263139
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,
E-mail:info@uou.ac.in
Website: https://www.uou.ac.in/